Optimal Sequence for Teamwork*

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Abstract

This paper analyzes a principal-agent model to study how the architecture of peer monitoring affects the optimal sequence of task assignment in teamwork. The agents work on a joint project, each responsible for an individual task. The principal determines the sequence of executing tasks as well as the rewards upon success of the project, the probability of which depends on each agent's effort and ability, with the objective of inducing full effort with minimum rewards. Agents may observe one another's effort based on an exogenous network as well as the endogenous sequence. We focus on networks composed of stars, and find a simple algorithm to characterize the optimal sequence of task assignment. The optimal sequence reflects the trade-off between the magnitude and the coverage of reward reduction in incentive design. In a single star, less capable periphery agents precede their center while more capable ones succeed their center. In complex networks consisting of multiple stars, periphery agents precede their center early in the sequence but succeed their center late in the sequence. When the number of peripheries differ across stars, a "V-shape" emerges: agents in large stars are allocated towards both ends of the sequence, while those in small ones towards the middle.

Keywords: Sequential task assignment, Peer information, Network, Incentive design

JEL Classification: D82, L14, L23

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1 Introduction

With the emergence of new business models and advanced communication technology, a large variety of workplace architectures have come into existence. Typical examples include the dense "open space" or "war room" models adopted by Bloomberg, Google, Goldman Sachs, etc., and in contrast loose-fitting designs such as Virtual Locations promoted by Amazon. Such an architecture essentially determines how information *can* flow internally among peers. For instance, a worker's job attitudes or efforts can be observed by neighboring colleagues in an open office, but not by someone remote or partitioned. As indicated by empirical evidence, workers' productivity and willingness to work respond positively to observed efforts of peers (Ichino and Maggi, 2000; Heywood and Jirjahn, 2004; Gould and Winter, 2009; Mas and Moretti, 2009). In this context, a principal responsible for incentive design is essentially endowed with a monitoring structure among agents that helps reduce the cost of inducing full effort: the more peers an agent can affect via a change in decision (e.g., from working to shirking), which means a greater impact on the success of the whole team, the less incentive needed for his effort exertion. However, there is still considerable room for designing the optimal incentives, as in many situations the principal can dictate the order of tasks, and hence how information *will* flow, as well as the outcome-based rewards. This paper seeks to answer a naturally spurred question in this context: given a workplace architecture, what is the optimal sequence of assigning tasks, and what is the optimal associated reward scheme?

Winter (2010) has provided a thorough discussion on the optimal incentive design under an exogenous task assignment sequence. In contrast, our paper focuses on the endogenization of the sequence in the principal's problem, and makes two contributions. First, we extend Winter's model to include the design of sequence as the principal's available option besides the reward scheme. Since there is a one-to-one relation between every sequence and the optimal rewards that follow, the optimal sequence predicted by our model is essentially unique for a wide range of architectures. Second, we explicitly characterize the optimal sequence for several typical classes of architectures, including simple ones such as cliques and stars, and composite ones such as core-periphery networks. We find that the solution relies heavily on the shape of the architecture as well as heterogeneity among agents. In general, the optimal sequence allocates agents with more transparent actions to intermediate positions, and less capable/important agents are assigned tasks earlier than their more capable/important peers.

Our model considers a group of multiple agents working on a joint project, where each agent is responsible for an individual task. Externally, the agents face the standard moral hazard problem: each agent needs to decide whether or not to exert effort, which cannot be observed by the principal, and the success probability of the whole project is determined by the joint effort profile. In most parts of our analysis, the agents' tasks are complementary. We allow agents to be heterogeneous, in the sense that some agent's effort may impose a greater influence on the success probability than others. Internally, the agents are connected to one another via an effort observation structure, which represents a workplace architecture in various applications. We model this structure as a network: a link ij between agent i and agent j implies that i observes j's effort if j makes his decision before i, and vice versa. In other words, feasibility of effort observation between two linked agents is bilateral in nature, while the actual flow of internal information is determined by the order of task execution.

The principal faces two problems of incentive design. First, she chooses and commits to a sequence of task assignment, in every period of which at least one agent is asked to make their decision on their corresponding task. Second, she offers a reward to each agent contingent on the final outcome of the project. The principal's objective is to minimize the total cost – that is, the sum of rewards upon success of the project – while inducing full effort from the agents.

Once the sequence is fixed, the network essentially produces a unique acyclic flow of internal information. We can then apply Winter (2010)'s result to characterize the optimal reward scheme. Also, a straight-forward application of results from other prior works (e.g. Winter (2006)) shows that the optimal sequence in a fully connected network, or a clique, is the reverse order in importance, i.e., the agent who is least influential to the project's success moves first, the second least influential agent moves second, and so forth. Our main analysis thus focuses on identifying the optimal sequence for more complex and possibly asymmetric networks.

Our first novel result characterizes the optimal sequence in a star network. Star networks are representative of many important workplace relationships and social interactions, for instance, the center being the general contractor for a construction project and the peripheries being the subcontractors who work on different parts of the building job and communicate with the general contractor. We propose a simple algorithm to find the optimal sequence, which is unique subject to a number of trivial variations with identical reward schemes. In the sequence, the center takes up a position between two subgroups of peripheries, allowing the principal to offer less incentives to peripheries before the center at the cost of more incentives to those after the center. The optimum represents the balance between marginal benefit and marginal cost. In addition, more important agents move later in the sequence, so that every agent after the center is more important than any agent before the center. Intuitively, when an early mover's decision affects a group of more important peers, his shirking will trigger a greater reduction in the project's success probability. Hence his implicit cost of shirking rises, which is always to the principal's advantage.

Not all workplace architectures or relationships can be approximated by simple structures such as cliques and stars. Instead, a complex architecture may be regarded as the composition of multiple simple ones, as in large projects that require the collaboration of several small teams. A typical class of such architectures is core-periphery networks, in which the centers of multiple stars are interlinked. We first consider a core-periphery structure for a vertical project, i.e. the order of executing tasks between stars is fixed while that within each star is decided by the principal. This architecture can represent a multi-phase project with vertical collaboration, e.g., the development of drugs that include preclinical, investigational and post-marketing phases. As above, we identify an algorithm that characterizes the essentially unique optimal sequence. In the sequence, one and only one of the stars plays a special role. Before it, all periphery agents of each star execute their tasks before their corresponding center, and obtain lower rewards when the project succeeds; after it, on the contrary, all periphery agents of each star execute their tasks after their corresponding center, and obtain higher rewards when the project succeeds. Such dichotomy results from monotonicity in the influence of an agent's action according to the position their star takes: the earlier their star is in the order, having the agent work his task before the center means more peers to be affected via internal information, and hence a higher benefit from reducing his reward for the principal; at the same time, fewer agents would affect his action if he were placed after the center, implying a lower opportunity cost from reducing the rewards of those agents.

We further turn our focus to a core-periphery network for a horizontal project, namely a set of inter-connected stars in which the order of executing tasks between stars is also decided by the principal. Projects that require horizontal collaboration of multiple departments, such as those for different components of an assembled final product like a cell phone or a motor vehicle, are typical examples of this architecture. Our result provides a partial characterization of the optimal sequence. On one hand, the sequence also features a "special" star before which all peripheries precede their center in task assignment and after which all peripheries follow their center. On the other hand, when the order of stars is left to the principal's discretion, we find that the number of peripheries in the optimal sequence must exhibit a "V-shape": before the "special" star, stars with more peripheries are assigned tasks first, while after the "special" star the pattern reverses. Placing a large star towards the end of the sequence allows the principal to raise the maximum size of reward reduction for each earlier agent, while placing one towards the beginning of the sequence exposes more agents to a larger-scale reward reduction.

The theoretical literature on incentive design for teamwork is extensive and growing. The trade-off an agent faces between working and shirking originated from the classical literature on moral hazard (Holmstrom, 1982; Holmstrom and Milgrom, 1991; Itoh, 1991). Subsequent studies developed this literature to static contracting on teamwork with a number of variations, such as externalities (McAfee and McMillan, 1991; Segal, 1999; Babaioff et al., 2012), specialization versus multitasking (Balmaceda, 2016), and loss-averse agents (Balmaceda, 2018). Che and Yoo (2001), Segal (2003), Bernstein and Winter (2012) studied contracting problems in a dynamic context, with the main focus on how various forms of externalities affect the optimal contracts. A comprehensive study on the role of internal information in teamwork, with an exogenous sequence of task assignment, has been given by Winter (2004, 2006, 2010). This is the main strand of literature that our work contrasts to by endogenizing the sequence. Finally, besides the empirical works mentioned above, experimental studies on behavior in team production have also indicated that an agent's contribution in teamwork is highly responsive to internal information (Carpenter et al., 2009; Steiger and Zultan, 2014) and that unequal rewards tend to facilitate coordination and improve efficiency (Goerg et al., 2010).

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 presents the result for a fully connected network, which is essentially a generalization and extension of existing theories. Section 4 and 5 present the results for a star network and a core-periphery network, respectively. Section 6 concludes. All omitted proofs are provided in the Appendix.

2 The Model

Players and actions. A principal (*she*) owns a project that is collectively managed by a set N of n agents. Each agent (*he*) is responsible for a single task and chooses whether to exert effort or not. Formally, each agent's action space is given by $A = \{0, 1\}$, with $a = 1$ if the agent chooses to exert effort and $a = 0$ if he shirks. The cost of effort is $c > 0$ and constant over all agents. To save on notation, we normalize c to 1 without loss of generality. Hereafter, we interchange the terms *work* and *exert effort*.

Network. The organizational structure, also referred to as the *network* of the agents, is characterized by an intrinsic and undirected graph g of unordered pairs (i, j) of agents who are directly linked. This network could originate from the workplace architecture, the authority structure, informal social networks and so forth. The structure of g is common knowledge.

Technology. The organization's technology is a mapping from a profile of effort decisions to a probability of the project's success. For a subset $S \subseteq N$ of working agents, the probability of the project's success is $p(S)$. Throughout the paper, we assume that p is increasing in the sense that if $T \subset S$, then $p(T) < p(S)$. Moreover, we distinguish between the technology's properties of complementarity and substitutability. A technology p satisfies *complementarity* among agents if for every two sets of agents S and T with $T \subset S$ and every agent $i \notin S$, we have $p(S \cup \{i\}) - p(S) > p(T \cup \{i\}) - p(T)$; that is, i's effort is more effective if the set of other agents who exert effort enlarges. By contrast, we say that p satisfies *substitutability* among agents if $p(S \cup \{i\}) - p(S) \leq p(T \cup \{i\}) - p(T)$. In addition, we distinguish between different agents' importances to the project. We say that agent i is (weakly) more important than j if for every coalition $S \subseteq N$ with $i, j \in S$, we have $p(S \backslash \{i\}) \leq p(S \backslash \{j\})$; that is, i's shirking is more detrimental than j's to the probability of success. We assume that the set N is totally ordered in terms of importance.

Mechanism. Before the agents execute their tasks, the principal has to choose a sequence of execution (permutation) π such that agent i is the π_i-th, with $\pi_i \in \{1, \dots, n\}$, player to act. In addition, the principal has to design a reward scheme $v = (v_1, \dots, v_n)$ such that agent i receives $v_i \geq 0$ if the project turns out to be successful, and receives zero payoff otherwise. A mechanism $\{\pi, v\}$ consists of a sequence of execution π and a reward scheme v for the agents. Throughout, we assume that the principal can commit to the mechanism.

Information. The principal cannot monitor the agents' effort choices, but knows whether the project is a success or a failure after all effort decisions have been made.

The agents' *internal information* about their peers' effort choices is jointly determined by the graph g and the permutation π. Specifically, agent i observes agent j's action, or simply i *sees* j, before i acts if and only if i and j are directly linked in g (i.e., $(i, j) \in g$), and i acts after j (i.e., $\pi_i > \pi_j$).[1] For every pair (g, π), we define $N_i(g, \pi) \coloneqq \{j | (i, j) \in g, \pi_i > \pi_j\}$ to be the set of agents whom agent i sees given the internal information determined by (g, π). To save on notation, we write N_i for the set $N_i(g, \pi)$ henceforth.

[1] If i and j act simultaneously, then neither of them can see the other.

Principal's problem. Consider the game that is defined by the set of agents N, the agents' action space A, the network g and a mechanism $\{\pi, v\}$. A strategy for agent i is a function $s_i : 2^{N_i} \to \{0,1\}$ which specifies the agent's effort choice as a function of his information on the effort decisions of other agents who are in N_i. For every strategy profile $s = (s_1, \dots, s_n)$, agent i's expected utility is given by

$$u_i(s) = p(W(s))v_i - s_i,$$

where $W(s)$ is the set of agents who work given s.

A mechanism $\{\pi, v\}$ is *effort-inducing (EFI)* with respect to the network if there exists a *perfect Bayesian equilibrium (PBE)* s^* for the associated game such that all the agents exert effort (i.e., $W(s^*) = N$). The principal's problem is to design an EFI mechanism that yields minimal total payoffs to the agents among all EFI mechanisms. We call this mechanism an optimal EFI mechanism. In particular, for a fixed permutation π, a reward scheme $v^*(\pi)$ is optimal if $\{\pi, v^*(\pi)\}$ is an optimal EFI mechanism. The principal's objective is economically meaningful when she has a relatively high value for the project and each agent is productive enough so that working increases the probability of a success significantly. Alternatively, one can consider the mechanism that maximizes the principal's net profit. We find this does not provide new insights while complicates the analysis remarkably.

Remark. Our model makes a notable assumption for the information structure, that is, an agent can observe only the actions of those who are directly connected to him. This stands in contrast to Winter (2006)'s model in which the agents can see all their predecessors' actions. Obviously, if an agent could observe each preceding action, then the information he possesses depends only on the sequence of execution, making the network irrelevant. Consequently, one would derive the same results as in Winter's model. In contrast, our assumption highlights the impacts of network topology on incentive design. It is reasonable when an individual has no convenient means to either send or receive information about others who are not spatially or socially close. For example, it might be difficult to provide hard evidence showing that a worker shirked; in many companies, it is unprofessional to discuss colleagues' job performances, preventing workers from sharing such information.

2.1 Preliminary Analysis

In this subsection, we follow Winter (2010) to characterize the optimal reward scheme $v^*(\pi)$ given a fixed sequence of execution π. We start the characterization with a complementary technology.

Let the technology p satisfies complementarity. Define $M_i(g, \pi)$, M_i for short, to be the set of agents satisfying that for each $j \in M_i$ there exists a sequence $\{k_r\}$ such that j sees k_1 sees k_2 sees $\ldots k_r$ sees i. That is, everyone in M_i can ultimately learn i's action if an agent could share his information with those who see him. Proposition 1 characterizes the optimal reward scheme $v^*(\pi)$ with respect to an arbitrary permutation π.

Proposition 1. *Suppose that p satisfies complementarity. For any fixed permutation π, the optimal reward scheme $v^*(\pi)$ exists and pays agent i $v_i^* = [p(N) - p(N \backslash (\{i\} \cup M_i))]^{-1}$.*

The intuition of Proposition 1 is that when the agents execute their tasks sequentially, they are facing an implicit threat of shirking; that is, the exposure of a low effort might induce an agent who observes this behavior to shirk and consequently trigger a *domino effect* of shirking, making a success less likely. This implicit threat reduces the agent's incentive cost. Under a complementary technology and an optimal reward scheme, it is indeed sequentially rational for an agent to shirk once he sees someone shirking, making the implicit threat credible. Moreover, Proposition 1 implies that if agent i's decision becomes more transparent in the sense that the set M_i increases, then the principal should pay i less since i has a greater implicit threat now. In general, an agent's decision is more transparent when he acts earlier in the sequence and has more links connected to him; thus, a worker in such position should expect lower remuneration.

We now turn to the case of substitutability. In contrast to the case of complementarity, under a substitutable technology, the internal information has no impact on incentives as if all the agents acted simultaneously. To implement full effort, the principal must provide the agents sufficient incentives when they believe that all their peers are working. However, due to the substitutability, such a reward scheme gives an agent an even stronger incentive to work when he sees someone shirking. This eliminates the implicit threat of shirking that is critical in the complementarity case, thereby preventing the principal from reducing the incentive costs. Formally, we have the following result:

Proposition 2. *Suppose that p satisfies substitutability. For every fixed permutation π, the optimal reward scheme $v^*(\pi)$ is identical and pays agent i $v_i^* = [p(N) - p(N \backslash \{i\})]^{-1}$.*

Proposition 2 indicates that peer information does not reduce incentive costs under a substitutable technology. This is because under an effort-inducing reward scheme the agents are incentivized to substitute own effort for those whom they see shirking. Thus, if an agent chooses to shirk, it does not affect the others' decisions. This means that an effort-inducing reward scheme has to provide an agent sufficient incentive when he believes that all his peers exert effort. It is tempting to note that such a reward scheme is also required by a full-effort Nash equilibrium of the game in which the agents choose their efforts simultaneously. Since the optimal reward scheme depends not on the information structure, every permutation is identical payoff-wise. Thus, in the subsequent sections, we assume that the technology satisfies complementarity. Since N is a finite set, Proposition 1 ensures that an optimal EFI mechanism exists; it thus remains to find one.

3 Fully Connected Network

As a benchmark, in this section, we study a fully connected network in which all the agents are interconnected. This network topology yields the richest transparency in the sense that under any sequence of execution, each agent except the first mover can observe all preceding actions. As Proposition 1 characterized the optimal reward scheme for an arbitrary sequence of execution π, it remains to seek for the optimal permutation π^*. Besides characterizing π^*, we offer more general results than Winter (2006) and identify several key properties of the optimal sequence and the associated reward scheme.

As a first step, we show that if two agents are linked in a network, then they cannot act simultaneously in the optimal sequence. This is summarized by Lemma 1 below:

Lemma 1. *For any two agents i and j such that $(i, j) \in g$, we have that $\pi_i^* \neq \pi_j^*$ in the optimal sequence π^*.*

Proof. Suppose not, then $\pi_i^* = \pi_j^*$. Consider a new permutation π' which differs from π^* only in that j acts in π' immediately after i and before all the agents who act after i in π^*; thus, $\pi_j' > \pi_j^*$ and $\pi_k' = \pi_k^*$ for any agent $k \neq j$. This implies that $N_j^* \subseteq N_j'$ and $M_j' = M_j^*$. Consider an agent $k \neq j$. Clearly, if $\pi_k^* > \pi_j^*$, then $M_k' = M_k^*$. If $\pi_k^* \leq \pi_j^*$, then we partition M_k^* into two parts: $M_{k\backslash j}^*$ and $M_k^* \backslash M_{k\backslash j}^*$, where $M_{k\backslash j}^*$ is the set of agents who will remain in M_k^* if all j's links are eliminated and the agents act in the order of π^*. Pick any agent $l \in M_k^*$. If $l \in M_{k\backslash j}^*$, then clearly he will remain in M_k' under π'. If $l \in M_k^* \backslash M_{k\backslash j}^*$, it must be that $l \in M_j^*$. Since $N_j^* \subseteq N_j'$ and $M_j' = M_j^*$, l will still remain in M_k', and thus, $M_k^* \subseteq M_k'$. In summary, for any agent $k \in N$, we have $M_k^* \subseteq M_k'$, meaning that $v_k^*(\pi') \leq v_k^*(\pi^*)$ due to Proposition 1. But since $(i, j) \in g$ and $\pi_i^* = \pi_j^*$, we have $M_i' \subset M_i^*$; thus, $v_i^*(\pi') < v_i^*(\pi^*)$. This means that the total payoffs to the agents are strictly lower under π' than under π^*, leading to a contradiction. Thus, the lemma is proven. □

Lemma 1 states that under complementarity, it is always suboptimal to make two linked agents acting simultaneously. This is because doing so reduces the transparency of actions, thereby mitigating the implicit threat of shirking and increasing incentive costs. Lemma 1 implies that in a fully connected network in which all the agents are interconnected, the optimal sequence is such that the agents act sequentially in the order $1, 2, \ldots, n$, though the specific order of each agent remains unknown. To fully characterize the optimal sequence, we relabel the agents in the way that agent i is (weakly) less important than $i+1$, $i \leq n-1$, with agent n being the most important. In addition, we say that agents i and j are *neighbors* if $(i,j) \in g$. Proposition 3 below shows that if two agents are neighbors and share the same neighbors other than themselves, then the optimal sequence satisfies that if one agent acts immediately after the other, then the more important agent acts later. This implies that the optimal sequence for a fully connected network is the identity permutation.

Proposition 3. *For any two agents i and j such that $(i,j) \in g$, $\{k|(i,k) \in g\}\backslash\{j\} = \{k|(j,k) \in g\}\backslash\{i\}$ and i is more important than j, if in the optimal sequence π^*, $|\pi_i^* - \pi_j^*| = 1$, then $\pi_i^* = \pi_j^* + 1$. Thus, if g is a fully connected network and the agents are increasingly important, then the optimal EFI mechanism $\{\pi^*, v^*\}$ satisfies: (i) π^* is the identity permutation; (ii) agent i receives payoff $v_i^* = [p(N) - p(\{j|j < i\})]^{-1}$. In particular, if two agents are equally important, then switching the two agents' orders in π^* still results in an optimal sequence and does not change the total payoffs.*

Proof. We first prove that $\pi_i^* = \pi_j^* + 1$. Suppose not, then $\pi_j^* = \pi_i^* + 1$. Now switch i and j and call the new permutation π'. Since $\{k|(i,k) \in g\}\backslash\{j\} = \{k|(j,k) \in g\}\backslash\{i\}$ and $\pi_j^* > \pi_i^*$, we have $N_i' \cup \{i\} = N_j^* \cup \{j\}$, $N_j' = N_i^*$, $M_i' = M_j^*$ and $M_j' \cup \{j\} = M_i^* \cup \{i\}$. Consider an agent $k \neq i, j$. By Lemma 1, for any of i's neighbors l, we have either $\pi_l^* < \pi_i^*$ or $\pi_l^* > \pi_j^*$. Thus, there are two possibilities to consider.[2] First, $i, j \notin M_k^*$. Since $N_i' \cup \{i\} = N_j^* \cup \{j\}$ and $N_j' = N_i^*$, the switch between i and j will not affect M_k; thus, $M_k' = M_k^*$, meaning that $v_k^*(\pi') = v_k^*(\pi^*)$. Second, $i \in M_k^*$, then $j \in M_k^*$. Since $M_i' = M_j^*$ and $M_j' \cup \{j\} = M_i^* \cup \{i\}$, the switch will not affect M_k; thus, $M_k' = M_k^*$ and $v_k^*(\pi') = v_k^*(\pi^*)$. Note that $M_i' = M_j^*$ and i is more important than j, thus we have

$$p(N\backslash(\{j\} \cup M_j^*)) = p((N\backslash M_j^*)\backslash\{j\}) > p((N\backslash M_j^*)\backslash\{i\}) = p(N\backslash(\{i\} \cup M_i')).$$

It follows from Proposition 1 that $v_i^*(\pi') < v_j^*(\pi^*)$. In addition, since $M_j' \cup \{j\} = M_i^* \cup \{i\}$,

[2] The case that $j \in M_k^*$ but $i \notin M_k^*$ is impossible. This is because if $j \in M_k^*$ then $\pi_k^* < \pi_i^*$, meaning that $i \in M_k^*$ as $\{k|(i,k) \in g\}\backslash\{j\} = \{k|(j,k) \in g\}\backslash\{i\}$, leading to a contradiction.

we have $v_j^*(\pi') = v_i^*(\pi^*)$. This implies that the total payoffs to the agents are strictly lower under π' than under π^*, leading to a contradiction.

The second part of the proposition is thus immediate. Note that if g is a fully connected network, then by Lemma 1, the agents act sequentially under π^*. If further the agents are increasingly important, then the above result implies that the optimal sequence is the identity permutation, and thus, the optimal reward scheme is given by Proposition 1 accordingly. Finally, if two agents are equally important, then it is readily confirmed that switching these agents does not affect any agent's incentive cost. Therefore, the proposition is proven. □

Proposition 3 suggests that in a fully connected network, the principal should delay more important tasks towards the end of the production process if feasible. Intuitively, when an agent shirks under the optimal reward scheme, he triggers all his successors to shirk. If the agent and his successors together are more important to the project, then he faces a greater implicit threat of shirking and is thus easier to be incentivized. Analogously, on the equilibrium path, agent i makes his decision as if he worked on an independent project; if he chooses to exert effort, then the project yields a high output equal to $p(N)v_i^*$, with v_i^* fixed; otherwise, the project yields a low output equal to $p(\{j|j < i\})v_i^*$. By allocating more important agents into later stages, the principal essentially reduces the low output level without changing the high output; clearly, the agent will be more willing to exert effort.

Propositions 3 implies that agents who are allocated to later stages under the optimal mechanism should be compensated more generously, even if all the agents are equally important in nature. The idea is particularly transparent for fully connected networks. That is, on the equilibrium path, each agent could alternatively free ride on his predecessors' efforts instead of exerting effort himself, while the optimal reward just offsets his free riding incentive; an agent who involves in a later stage can free ride on more predecessors' efforts, and thus, his incentive cost is larger. Moreover, due to complementarity, the gap between two adjacent agents' rewards increases in their orders. This is because complementarity corresponds to an increasing return-to-scale technology; as more agents exerted efforts, the free riding behavior becomes more detrimental, and thus, the principal incurs increasingly more incentive costs to induce effort. These results are summarized by the following corollary:

Corollary 1. *Suppose that g is a fully connected network, then in the optimal reward scheme v^*: v_i^* is increasing and strictly convex in i under the optimal sequence π^*.*

Proof. From Proposition 3, the gap between v^*_{i+1} and v^*_i is given by

$$v^*_{i+1} - v^*_i = \frac{p(\{j|j < i+1\}) - p(\{j|j < i\})}{[p(N) - p(\{j|j < i+1\})][p(N) - p(\{j|j < i\})]}.$$

The numerator of the right-hand side (RHS) is increasing in i due to the complementarity of p; the denominator is decreasing in i due to the monotonicity of p, and thus, $v^*_{i+1} - v^*_i$ is increasing in i. The monotonicity of v^*_i follows immediately from Proposition 3. □

Since a fully connected network yields the richest transparency, it can impose the greatest implicit threat of shirking on the agents. The corollary below states that the total payoffs to the agents are the least in fully connected networks among all network topologies. Thus, Propositions 3 provides a sharp lower bound for the total payoffs of optimal EFI mechanisms.

Corollary 2. *A fully connected network yields minimal total payoffs to the agents, and thus maximal payoff to the principal.*

Proof. Let g_1 be a fully connected network and g_2 be an arbitrary network with the same amount of vertices as g_1. From Lemma 1, a permutation with some simultaneous moves is weakly suboptimal for g_2.[3] Without loss, assume that in the optimal sequence $\pi^*(g_2)$ of g_2 the agents act sequentially in the order $1, 2, \ldots, n$. Consider the permutation $\pi(g_1)$ of g_1 such that each agent has the same order in $\pi(g_1)$ as in $\pi^*(g_2)$. Clearly, for each agent i, M_i is weakly larger under $\pi(g_1)$. Then from Proposition 1, the optimal reward scheme pays i less under $\pi(g_1)$. Since $\pi(g_1)$ is not necessarily optimal, the optimal total payoffs must be (weakly) lower under g_1. Thus, the corollary is proven. □

Corollary 2 indicates that a fully connected network is the best network topology for the principal as it yields the richest internal information. Such a network can represent the emerging workplace architecture "war room" that is implemented by different organizations. The movement to such open-space environment allows workers to monitor their peers more easily, making the peer information about effort more transparent. Consequently, it enhances the implicit incentive of working that is imposed by this mutual observability. In our model, the network g is exogenously given; otherwise, if the principal can improve the connection between agents (i.e., by adding links to g) at relatively low costs, she may find it profitable to transform g into a fully connected network.

[3]Note that if agent i is not linked to anyone else, then his incentive cost is fixed at $[p(N) - p(N\backslash\{i\})]^{-1}$. Thus, whether there is another agent acting simultaneously does not affect i's incentive cost.

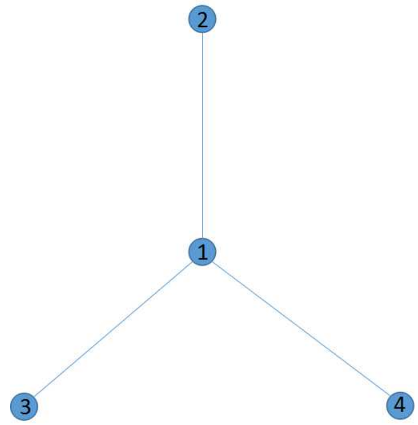

Figure 1: An Example of Star Network.

4 Star Network

In this section and the next, we study the optimal sequence for several typical network topologies, starting with star networks. A star network features a particular node i such that every link in the network involves node i; thus, agent i is termed as the *center* of the star, and the rest of the agents are termed as the *peripheries* of the star. The layout of a star network is depicted in Figure 1. In this section, we assume that $n \geq 3$.

Star network structures are common in organizations. For example, in most scientific labs, when a project leader coordinates with his/her fellow researchers, the leader often works as the center of the team, while each fellow researcher works on an individual task and communicates the progress to the project leader. Such a team thus has a star network structure. Usually, the principal investigator (PI) of the lab can only observe the outcome of the entire project and chooses how to reward the team based on the outcome. In large-scale constructions, a general contractor who is in charge of the overall coordination of a project performs part of the building work and contracts subcontractors to perform specific and independent tasks. Such a team also can be represented by a star network, with the general contractor being the center and the subcontractors being the peripheries. In most cases, the owner/developer (principal) observes the quality of the entire building during the inspection phase and remunerates the contractors based on the quality.

To find the optimal sequence for a star network, it suffices to characterize the set of the center's successor(s), with the possibility of an empty set. For ease of exposition, we relabel the peripheries by importance from 1 to $n-1$, with a higher index referring to a more important agent. Provided there is no confusion, let the center be the n-th agent who is not necessarily the most important. As a useful general result, Lemma 2 below shows that if two agents share the same nonempty set of neighbors, then the optimal sequence satisfies that if the two agents have a neighbor who acts between them, then the more important one of the two agents acts in a later stage than the other.

Lemma 2. *For any two agents i and j such that $\{k|(i,k) \in g\} = \{k|(j,k) \in g\} \neq \emptyset$ and i is more important than j, if in the optimal sequence π^*, there exists some $k' \in \{k|(i,k) \in g\}$ such that $\pi_i^* \wedge \pi_j^* < \pi_{k'}^* < \pi_i^* \vee \pi_j^*$, then $\pi_i^* > \pi_j^*$.*

The proof of Lemma 2 is analogous to that of Proposition 3. Lemma 2 implies that if in the optimal sequence the center has a nonempty set of predecessors and successors, respectively, then the center's successors are uniformly more important than his predecessors. The intuition has been suggested previously; that is, if more important agents act in later stages, then one's shirking will induce agents with higher importance to shirk and is thus more detrimental to the probability of success, thereby allowing the principal to reduce incentive costs. The relative importance between the center's predecessors and successors implies that the optimal sequence for a star network can be summarized by a sufficient statistic, that is, the number of the center's successor(s).[4] Let m be the number of the center's successor(s), with $0 \leq m \leq n-1$. Then, the center has $n-1-m$ predecessors; if each of them shirks, then the center and all his successors shirk correspondingly. Similarly, if the center shirks, then all his successors shirk too. However, the center's successors cannot trigger other's shirking, since their actions are unobservable. Thus, from Proposition 1, the total payoffs to the agents under an optimal reward scheme v^* is given by

$$v^*(m) = \underbrace{\sum_{i=1}^{n-1-m} \frac{1}{p(N) - p(\{j|j < n-m\} \backslash \{i\})}}_{\text{payoffs to the predecessors}} + \underbrace{\frac{1}{p(N) - p(\{j|j < n-m\})}}_{\text{payoff to the center}} + \underbrace{\sum_{n-m}^{n-1} \frac{1}{p(N) - p(N \backslash \{i\})}}_{\text{payoffs to the successors}}.$$

To find the optimizer m^*, we compare $v^*(m)$ with $v^*(m+1)$; the difference between the two items is the marginal effects of increasing the center's successors on the total rewards.

[4] This is because the relative orders between the center's predecessors or successors does not affect their incentive costs, as each individual's action is equally transparent for predecessors and successors, respectively.

By preliminary calculation, for any m with $0 \leq m \leq n-2$, we have

$$\begin{aligned} v^*(m+1) - v^*(m) = &\sum_{i=1}^{n-2-m} \frac{1}{p(N) - p(\{j|j < n-m-1\} \backslash \{i\})} \\ &- \sum_{i=1}^{n-2-m} \frac{1}{p(N) - p(\{j|j < n-m\} \backslash \{i\})} \\ &- \frac{1}{p(N) - p(\{j|j < n-m\})} + \frac{1}{p(N) - p(N \backslash \{n-m-1\})}. \end{aligned} \tag{1}$$

The sum of the first three terms on the RHS of (1) is the net change in payoffs to the center and his predecessors. Since p is increasing, this value is negative, i.e., by increasing the center's successors, the total payoffs to the center and his predecessors decrease. The reason is twofold: first, increasing the center's successor reduces the number of the rest of the agents; more important, doing so makes the efforts of the center and his predecessors more transparent, thereby enhancing the implicit threat of shirking for these agents and reducing the incentive costs. In this regard, we call these terms together the marginal benefit (MB) of increasing the center's successors. Formally, we define

$$\begin{aligned} MB(m) := &\sum_{i=1}^{n-2-m} \left[\frac{1}{p(N) - p(\{j|j < n-m\} \backslash \{i\})} - \frac{1}{p(N) - p(\{j|j < n-m-1\} \backslash \{i\})} \right] \\ &+ \frac{1}{p(N) - p(\{j|j < n-m\})}. \end{aligned}$$

In contrast, the last term on the RHS of (1) is positive, which is the extra reward to the new successor. Analogously, we call this term the marginal cost (MC) of increasing the center's successors. Formally, we define,

$$MC(m) := \frac{1}{p(N) - p(N \backslash \{n-m-1\})}.$$

Note that $MC(m)$ is non-decreasing in m. This is because each new successor of the center is (weakly) less important than the current ones, and thus, his incentive cost is higher. It follows that if $MB(m)$ is decreasing in m, then there exists a unique optimizer m^* (either an interior solution or a corner solution). Lemma 3 below shows that under complementarity, $MB(m)$ is indeed decreasing in m.

Lemma 3. *$MB(m)$ is decreasing in m.*

The idea is that on the equilibrium path, the center and his predecessor could alternatively free ride on the other predecessors' efforts. As the center obtains more successors, there are fewer agents whose effort one can free ride on. Since complementarity corresponds to an increasing return-to-scale technology, free riding becomes less detrimental as the number of the center's successors rises. Furthermore, since the center's successors are uniformly more important than the predecessors as required by the optimal sequence, the shirking behaviors of the center and his predecessors will trigger on average less important agents to shirk as the center obtains more successors, meaning that the average implicit threat of shirking is relatively weak. In summary, on both the extensive and intensive margin, increasing the center's successors becomes less effective in reducing the incentive costs of the center and his predecessors as the number of the center's successors increases.

Lemma 3 ensures that the optimal sequence is essentially unique and can be succinctly characterized by an integer m^* which is the smallest m such that $MB(m) \leq MC(m)$. The next proposition shows that in the optimal sequence, the center never acts the first; if the center is sufficiently more important than all the peripheries, then he acts the last.

Proposition 4. *Suppose that g is a star network with $n \geq 3$ agents, then the optimal EFI mechanism $\{\pi^*, v^*\}$ satisfies: (i) the center has m^* successor(s) with $0 \leq m^* \leq n-2$ and each of them is more important than all the center's predecessors; (ii) the optimal reward scheme v^* is characterized by Proposition 1 accordingly. Furthermore, if $[p(N) - p(N\backslash\{n-1\})] < \delta\,[p(N) - p(N\backslash\{n\})]$ for some small $\delta > 0$, then $m^* = 0$, where agent $n-1$ is the most important periphery agent and agent n is the center.*

Proof. The optimal sequence π^* is given by Lemmas 2 and 3. Indeed, Lemma 3 implies that $m^* = \min\{m | MB(m) \leq MC(m)\}$. Then, the optimal reward scheme v^* is characterized by Proposition 1 accordingly. To see that m^* is bounded above by $n-2$, note that

$$MB(n-2) = \frac{1}{p(N) - p(\{1\})} < \frac{1}{p(N) - p(N\backslash\{1\})} = MC(n-2),$$

and thus, $m^* \leq n-2$. To prove the last statement of the proposition, note that

$$MC(0) = \frac{1}{p(N) - p(N\backslash\{n-1\})},$$

and that

$$
\begin{aligned}
MB(0) &= \sum_{i=1}^{n-2} \left[\frac{1}{p(N) - p(\{j|j < n\} \backslash \{i\})} - \frac{1}{p(N) - p(\{j|j < n-1\} \backslash \{i\})} \right] \\
&\quad + \frac{1}{p(N) - p(N \backslash \{n\})} \\
&< \sum_{i=1}^{n-2} \frac{1}{p(N) - p(\{j|j < n\} \backslash \{i\})} + \frac{1}{p(N) - p(N \backslash \{n\})} \\
&< \frac{n-1}{p(N) - p(N \backslash \{n\})}.
\end{aligned}
$$

Let $\delta = \frac{1}{n}$; thus, if $[p(N) - p(N\backslash\{n-1\})] < \delta\,[p(N) - p(N\backslash\{n\})]$, then $MB(0) < MC(0)$. This implies that $m^* = 0$. The proposition is thus proven. □

The intuition of why the center never acts the first is straightforward. Note that for star networks, the first mover's incentive cost is constant, as his shirking always induces everyone to shirk. Rather than making the center the first mover, letting one periphery acts the first will lead to one more agent whose action can be observed by others, thereby improving transparency and reducing incentive costs. Hence, the optimal sequence requires that the center never acts the first.

Proposition 4 states that if the center is sufficiently more important than the peripheries, in the sense that the center's shirking is much more detrimental to the probability of success, then the center should act the last. Intuitively, if the center is relatively more important, then his predecessors' incentive costs are relatively low due to the large implicit threat of shirking. In contrast, the center's successors do not have such implicit threat and thus have relative high incentive costs.[5] Consequently, increasing the center's successors is unprofitable.

Remark. In particular, if all the agents are equally important, then it can be easily proven that $1 \leq m^* \leq n-2$; that is, the center always acts in an interior stage. This is because if the agents are equally important, then each agent i's incentive cost depends only on the cardinality of M_i, irrespective of his identity. By allocating the center into an interior stage, the mechanism allows the periphery agents to learn their peers' actions through the center, as if the center acted as an internal communication device. This makes the agents' actions more transparent, thereby reducing the incentive costs.

[5] Indeed, for any $1 \leq m \leq n-2$, the incentive costs of the center and his predecessors' are bounded above by $[p(N) - p(N\backslash\{n\})]^{-1}$, whereas that of a center's successor is bounded below by $[p(N) - p(N\backslash\{n-1\})]^{-1}$. By assumption, the latter incentive cost is more than $1/\delta$ times of the former, for some small $\delta > 0$.

The previous analysis indicates that there exists a simple algorithm to find the optimal sequence for star networks. Specifically, one just needs to allocate the peripheries into the set of the center's successors one by one from the most important to the least, until the first time when $MB(m) \leq MC(m)$. In practice, this process is remarkably simpler than searching the optimal sequence for a general network topology. Moreover, the algorithm remains valid even if the relative order between the center and some peripheries is nonadjustable. This can be achieved by applying the original algorithm to the remaining peripheries. See Appendix A.2 for further details.

As a comparative-statics analysis, we study the impacts of the importance of individual task on the optimal sequence for star networks. Specifically, we examine how the number of the center's successors in the optimal sequence varies with the importance of individual task. For ease of exposition, in the following, we consider a numerical example and assume that the agents are equally important to the project.

Example 1. Suppose that g is a star network with $n \geq 3$ agents, and that the project is a success if and only if all tasks are successful. Each task is successful with probability 1 if the agent works, and is successful with probability $\alpha \in (0, 1)$ if the agent shirks. Hence, a lower probability α means that the failure of an individual task has critical implications on the entire project. Let w be the number of agents who work, then $p(w) = \alpha^{n-w}$ because all tasks are independent. Clearly, p is increasing and satisfies complementarity. Applying the previous results, we express $MB(m)$ and $MC(m)$ explicitly in the following:

$$\begin{aligned} MB(m, \alpha) &= \frac{n-m-2}{1-\alpha^{m+2}} - \frac{n-m-2}{1-\alpha^{m+3}} + \frac{1}{1-\alpha^{m+1}}, \\ MC(m, \alpha) &= \frac{1}{1-\alpha}. \end{aligned}$$

It follows that for fixed $\alpha \in (0, 1)$, $MB(m)$ is decreasing in m, and that $MB(0) > MC(0)$ and $MB(n-2) < MC(n-2)$. Thus, the optimizer m^* exists and is an interior solution for any $\alpha \in (0, 1)$. In addition, from basic mathematical analysis, we have that for fixed m, both $MB(\alpha)$ and $MC(\alpha)$ are increasing and strictly convex in α, and that $MB(\alpha)$ is single-crossing $MC(\alpha)$ from below in the domain $\alpha \in (0, 1)$. This is illustrated in Figure 2. It thus follows that the optimizer $m^*(\alpha)$ is non-decreasing in α.[6] Formally, we have:

Corollary 3. *In the optimal sequence π^*, the number of the center's successors $m^*(\alpha)$ is non-decreasing in α for $\alpha \in (0, 1)$.*

[6] Since m is an integer, $m^*(\alpha)$ is not necessarily increasing in α.

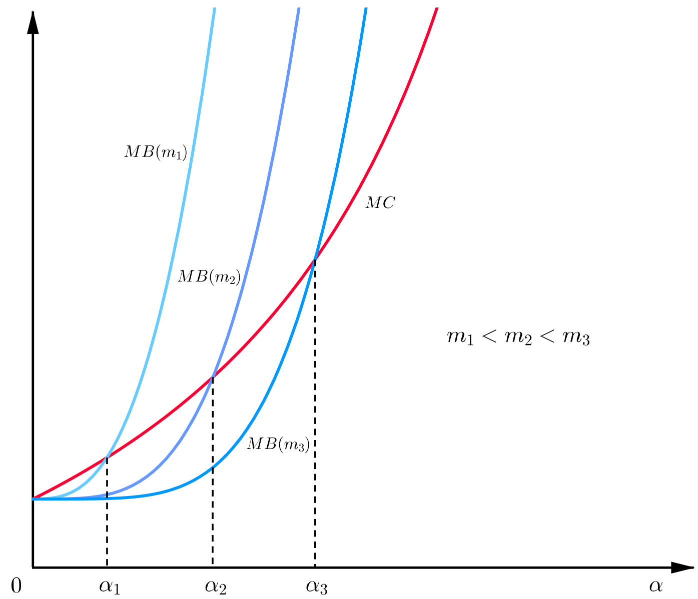

Figure 2: Marginal Benefit and Marginal Cost as a Function of Importance.

Corollary 3 implies that the more important each task is, the fewer successors the center has in the optimal sequence. Intuitively, if each task is important to the project's success, then each agent has a relatively strong incentive to work. Thus, the implicit threat of shirking is not crucial in providing incentive. This means that improving the transparency of actions by increasing the center's successors is not effective in reducing incentive costs. In contrast, if each individual task has little effect on the project's success, then each agent has a relatively strong incentive to shirk. In this case, the implicit threat of shirking plays an important role in providing incentive. Thus, the principal should make shirking behaviors more transparent by increasing the center's successors, thereby enhancing the implicit threat of shirking.

5 Core-Periphery Network

In Sections 3 and 4, we studied two simple network topologies, fully connected networks and star networks. However, not all organizational structures can be approximated by such simple networks; rather, in many organizations, a more complex structure might emerge as a composition of multiple simple ones. For example, in large projects that require the collaboration of several teams, the organizational structure can be represented by a network composed of multiple stars. In this section, we study a typical class of such networks – core-periphery networks, in which the centers of multiple stars are interconnected. The layout of a core-periphery network is illustrated in Figure 3.

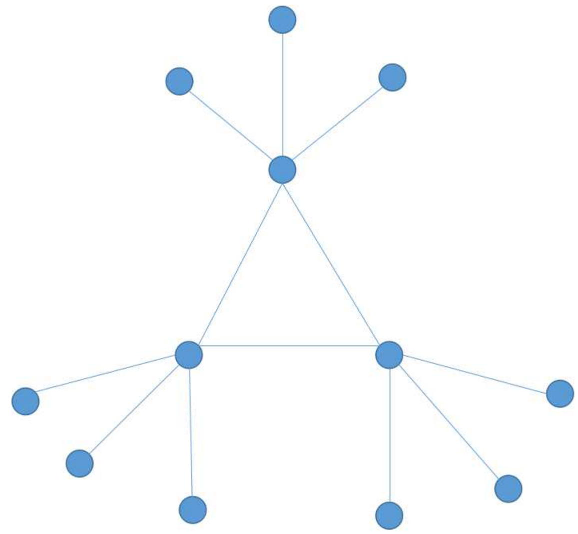

Figure 3: An Example of Core-Periphery Network.

Based on the nature of production process, we consider two cases. In a *vertical* project, the relative order between different stars is fixed while that of the agents within each star is determined by the principal. This feature can represent a multi-phase project with vertical collaboration, such as the development of drugs that includes preclinical, investigational and post-marketing phases. The orders of different phases cannot be interchanged. In contrast, in a *horizontal* project, the principal can additionally determine the relative order between different stars. Projects that require horizontal collaboration of multiple departments such as those for different components of an assembled final product (e.g., a cell phone and a motor vehicle) typically have such feature. For tractability, we assume throughout this section that all the agents are equally important to the project. This implies that for any subset W of working agents, $p(W)$ can essentially be replaced by $p(|W|)$ with a slight abuse of notations, where $|W|$ is the cardinality of set W.

We first consider a vertical project. Let r be the number of stars in the network g. For ease of exposition, we assume that all stars act sequentially, and label the stars by their relative order such that each agent in star i acts before any agent in star $i+1$. Let s_i be the number of agents in star i; thus, we have $\sum_{i=1}^{r} s_i = n$. The principal's problem is to choose the sequence of execution within each star separately. Since the relative order between different stars is fixed, the internal sequence of each star jointly determines the sequence of execution for the entire project. Note that given the sequence, the core-periphery network yields the same transparency as the network in which the centers of the stars are linked in a chain according to their relative orders (see Figure 4). This is because for each agent i, M_i is identical between the two networks under the same sequence of execution.

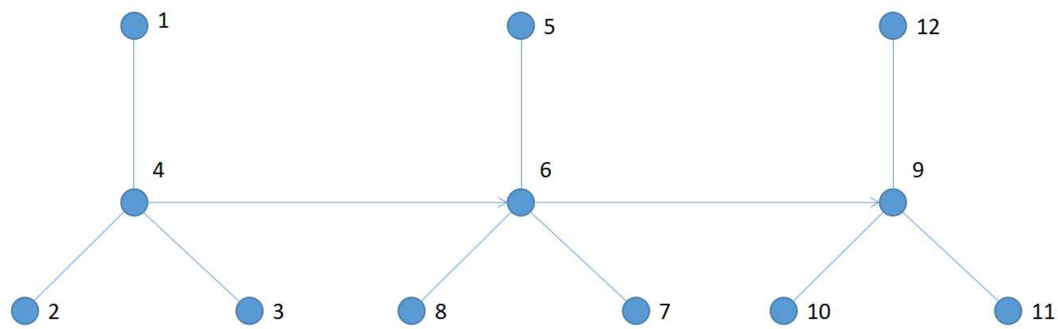

Figure 4: A Vertical Project Conducted in a Connected-Stars Network. In this sequence, the relative orders between different stars is fixed. Each agent acts before every agent of a star to his right. The label of each node denotes the order of the agent in the entire sequence. Given the sequence, such a network yields the same transparency as the core-periphery network in Figure 3.

To find the optimal sequence, we introduce, analogously to star networks, the marginal benefit and marginal cost of increasing the center's successors within a single star, namely, the marginal effects of allocating one more periphery to the set of the center's successors. To be succinct, we abuse the notations a little and again use the terms MB for marginal benefit and MC for marginal cost. As explained previously, the marginal benefit stems from the improvement of the transparency of preceding actions as well as the decrease in the number of the center's predecessors; the marginal cost is simply the extra payoff to the new successor. Since the agents are equally important, each agent i's incentive cost depends only on the cardinality of M_i. Thus, within any specific star, if a periphery acts after the center, then his incentive costs is fixed at $[p(n) - p(n-1)]^{-1}$. This implies that the marginal cost is constant across stars. The marginal benefit, in contrast, has an inter-star effect; that is, for any star except the first, increasing the center's successors within this star will improve the transparency of not only the center's action, but also all preceding actions that the center can learn on equilibrium path. To illustrate, consider the example in Figure 4. If alternatively agent 5 acts as agent 6's successor, then not only agent 6's action becomes more transparent, but also agent 1 to 4's actions are more transparent too, as these actions can be learned by agent 6 on the equilibrium path. Indeed, by moving agent 5 after 6, $|M_i|$ increases by 1, for $i = 1, 2, 3, 4$ and 6, and thus, v_i^* becomes lower.

Analogously to star networks, let m_i be the number of peripheries who act after the center within star i, with $0 \leq m_i \leq s_i - 1$; thus, $MB(m_i)$ stands for the marginal benefit of allocating one more periphery of star i into the set of its center's successors; the marginal cost MC is fixed at $[p(n) - p(n-1)]^{-1}$. We first establish a useful result by Lemma 4 below. It indicates that if in the optimal sequence a center acts after some of his peripheries, then all preceding centers act the last within own star; if in contrast, a center acts before some of his peripheries, then all subsequent centers act the first within own star.

Lemma 4. *Suppose that the project is vertical, then the optimal sequence π^* satisfies: (i) if there exists a star i, with $i > 1$, such that $s_i - 1 - m_i^* > 0$, then for any star $j < i$, $m_j^* = 0$; (ii) if there exists a star i, with $i < r$, such that $m_i^* > 0$, then for any star $j > i$, $m_j^* = s_j - 1$.*

Proof. We first prove statement (i). Suppose that in the optimal sequence, there exist two stars i and j, with $i > j$, such that $s_i - 1 - m_i^* > 0$ and $m_j^* > 0$, then we must have $MB(m_i^*) \leq MC \leq MB(m_j^* - 1)$; otherwise, the principal can make a locally profitable deviation by raising m_i^* by 1 or reducing m_j^* by 1. On the other hand, we must also have $MB(m_i^*) > MB(m_j^* - 1)$. To see this, let K_1 be the set of agents such that for any agent $k_1 \in K_1$, if m_i^* increases then $|M_{k_1}|$ increases. Similarly, let K_2 be the set of agents such that for any agent $k_2 \in K_2$, if m_j^* increases (equivalently, $m_j^* - 1$ increases) then $|M_{k_2}|$ increases. Given the organizational structure, since $i > j$, we must have $K_2 \subset K_1$. This implies that $MB(m_i^*) > MB(m_j^* - 1)$, leading to a contradiction. The proof of statement (ii) is analogous. Thus, the lemma is proven. □

The idea of Lemma 4 is straightforward. Since the stars' centers are ordered in a chain, the marginal benefit of increasing the center's successors within a star is always higher than that of a star in an earlier stage, as actions in later stages can impose the implicit threat of shirking on more preceding agents. Thus, if a center does not act the first within own star, then we have that the marginal benefit of increasing this center's successors is less than the marginal benefit. This implies that all preceding centers should act the last within own star, as the corresponding marginal benefits are even lower. Analogously, if a center does not act the last within own star, then when he has fewer successors the marginal benefit is higher than the marginal cost, meaning that all subsequent centers should act the first within own stars, as the corresponding marginal benefit are even higher.

Note that in any sequence, each star can only have three possible internal sequences. We say that a star is a *type I* star if the center acts the last within own star, is a *type II* star if the center acts in an interior stage within own star, and is a *type III* star if the center acts the first within own star. Lemma 4 implies that in the optimal sequence, there can be at most one type II star, and all preceding stars (if any) should be type I and all subsequent stars (if any) should be type III. The other possible case is that there are several type I stars followed by type III stars, with the possibility that there are only type I stars or type III stars.

Consequently, we could establish a simple algorithm to find the optimal sequence for a core-periphery network under a vertical project. Specifically, we first assume that all stars are type I stars. Then, from the last star to the first, we allocate the peripheries one by one into the set of the center's successors. Note that the optimal sequence must emerge in some stage of this process. Thus, if the marginal benefit is non-increasing through this process, then the optimal sequence is obtained once $MB \leq MC$. The next proposition shows that such an algorithm is indeed valid.

Proposition 5. *Suppose that the project is vertical, then the optimal sequence π^* can be obtained through the following procedure: first, make each star a type I star; second, from the last star to the first, allocate the peripheries one by one into the set of the center's successors, until $MB(m_i^*) \leq MC$ for some star i in which m_i^* peripheries act after the center. In the optimal sequence, no center acts the first or the last in the entire sequence. The optimal reward scheme v^* is given by Proposition 1 accordingly.*

The algorithm characterized by Proposition 5 again allows us to find the optimal sequence in a monotonic way, which remarkably simplifies the searching process. However, we must point out that this simple algorithm relies on the assumption that the agents are equally important. If the agents are differently important, then the algorithm might not hold, as the marginal cost is not necessarily monotone across stars.

We now turn to a horizontal project. The only difference is that the principal is now able to choose the relative order between different stars, while an agent of a star in an earlier stage still acts before every agent of a subsequent star. Moreover, Lemma 1 implies that in the optimal sequence all stars should act sequentially, as simultaneous moves reduce the transparency. Thus, once the relative order between stars is determined, the rest of the analysis is identical to a vertical project. Although we are unable to fully characterize an algorithm to pinpoint the optimal sequence for a horizontal project, we find a useful property of the optimal sequence which can remarkably simplify the searching process. This is summarized by the proposition below.

Proposition 6. *Suppose that the project is horizontal, then the optimal sequence π^* satisfies: for any $1 \leq i \leq r-1$, (i) if both stars i and $i+1$ are type I stars, then $s_i \geq s_{i+1}$; (ii) if both stars i and $i+1$ are type III stars, then $s_i \leq s_{i+1}$.*

Proof. We first prove statement (i). Suppose not, then $s_{i+1} > s_i$. Thus, for any periphery j of star i and any periphery k of star $i+1$, we have $|M_j^*| = |M_k^*| + 1$. Now switch star i and $i+1$ with both stars remained as type I. Call this new permutation π'. Note that after the switch, $|M_j'| = |M_k^*|$ and $|M_k'| = |M_j^*|$, whereas $|M_l'| = |M_l^*|$ for any agent l who is not a periphery of either star i or $i+1$. From Proposition 1, the difference in total payoffs equals

$$\begin{aligned}
v^*(\pi^*) - v^*(\pi') = &\left[\frac{s_i - 1}{p(n) - p(n-1-|M_j^*|)} + \frac{s_{i+1} - 1}{p(n) - p(n-1-|M_k^*|)}\right] \\
&- \left[\frac{s_i - 1}{p(n) - p(n-1-|M_j'|)} + \frac{s_{i+1} - 1}{p(n) - p(n-1-|M_k'|)}\right] \\
=&\frac{s_{i+1} - s_i}{p(n) - p(n-1-|M_k^*|)} - \frac{s_{i+1} - s_i}{p(n) - p(n-1-|M_j^*|)} \\
>&0.
\end{aligned}$$

This implies that π^* is not an optimal sequence, leading to a contradiction.

Then, we prove statement (ii). Suppose not, then $s_i > s_{i+1}$. Let agent j be the center of star i and agent k be the center of star $i+1$. Thus, we have $|M_j^*| = |M_k^*| + s_i$. Now switch star i and $i+1$ with both stars remained as type III. Call this new permutation π'. Note that after the switch, $|M_j'| = |M_k^*| - s_{i+1} + s_i$ and $|M_k'| = |M_j^*|$, whereas $|M_l'| = |M_l^*|$ for any agent l who is not the center of either star i or $i+1$. From Proposition 1, the difference in total payoffs equals

$$\begin{aligned}
v^*(\pi^*) - v^*(\pi') = &\left[\frac{1}{p(n) - p(n-1-|M_j^*|)} + \frac{1}{p(n) - p(n-1-|M_k^*|)}\right] \\
&- \left[\frac{1}{p(n) - p(n-1-|M_j'|)} + \frac{1}{p(n) - p(n-1-|M_k'|)}\right] \\
=&\frac{1}{p(n) - p(n-1-|M_k^*|)} - \frac{1}{p(n) - p(n-1-|M_j'|)} \\
=&\frac{1}{p(n) - p(n-1-|M_k^*|)} - \frac{1}{p(n) - p(n-1-|M_k^*| - s_i + s_{i+1})} \\
>&0.
\end{aligned}$$

The inequality is due to that $s_i > s_{i+1}$. This implies that π^* is not an optimal sequence, leading to a contradiction. Thus, the proposition is proven. □

Proposition 6 indicates that in the optimal sequence, if multiple consecutive stars are all type I stars, then a star with more agents is allocated in an earlier stage; in contrast, if these stars are all type III stars, then a star with more agents is allocated in a later stage. Thus, if in the optimal sequence both type I and type III stars are present, including the case that these two types are connected by a single type II star, then the permutation is "V-shaped" in terms of the number of agents within each star. Specifically, starting from the very beginning, we first observe a series of type I stars with the number of agents within each star decreasing.

Then, there may or may not be a single type II star which does not necessarily have fewer agents than previous stars. Finally, we observe a series of type III stars with the number of agents within each star increasing. While not offering a full characterization of the optimal permutation, Proposition 6 serves to exclude most suboptimal ones.

6 Conclusion

In this paper, we have characterized the optimal effort-inducing mechanism for teamwork with network-based internal information for typical networks composed of stars. Our model highlights the endogeneity of the task assignment sequence, and our results provide a simple algorithm to derive the optimal sequence. An agent's position is tightly related to his importance to the project as well as his connectivity in the network. More important agents move later in the sequence and receive higher rewards, while better connected agents take up intermediate positions, reflecting a balance between incentives for earlier and later agents. The general question of how to fully characterize the optimal incentive scheme in an arbitrary network remains open; richer studies in this direction may shed more light on incentive design in many contemporary circumstances with complex channels of internal information.

A Appendix

A.1 Omitted Proofs

Proof of Proposition 1.

Proof. We first prove that $\{\pi, v^*(\pi)\}$ is an EFI mechanism. Consider a strategy profile s^* such that $s_i^* = 1$ if and only if $a_j = 1$ for all $j \in N_i$ or N_i is empty; that is, an agent works unless he sees someone shirking. This strategy profile can be sustained by a PBE with the following set of beliefs: if $a_j = 1$ for all $j \in N_i$ or $N_i = \emptyset$, then $a_k = 1$ for all $k \in N \backslash (N_i \cup \{i\} \cup M_i)$; that is, an agent, not seeing anyone shirking, believes that those whom he cannot see and who cannot see him through a sequence of agents will exert effort. To verify this statement, note that if agent i shirks then by induction every $j \in M_i$ shirks as well. In contrast, if i works then he believes that all the other agents work too unless he sees someone shirking. Suppose i is the first to act, then he believes that if he works then all the other agents also work, and if he shirks then he will induce each agent in M_i to shirk. Thus, i prefers working to shirking if and only if the difference in expected reward exceeds the effort cost, i.e.,

$$[p(N) - p(N \backslash (\{i\} \cup M_i))]v_i \geq 1 \tag{A.1}$$

Clearly, v_i^* satisfies (A.1). It follows by induction that for all $\pi_i \in \{2, \ldots, n\}$, i prefers to work on equilibrium path if and only if (A.1) holds, as he sees no one shirking. Off the path, if i sees a nonempty subset $S_i \subseteq N_i$ of agents shirking, then he knows that each $j \in S_i$ will induce everyone in M_j to shirk. Let $R_i \coloneqq \bigcup_{j \in S_i} M_j \cup S_i$ be the set of agents whom i believes shirk. Thus, if i works then his expected utility equals $p(N \backslash R_i)v_i^* - 1$. In contrast, if i shirks then his expected utility equals $p((N \backslash R_i) \backslash (\{i\} \cup M_i))v_i^*$. We now provide a useful lemma.

Lemma A.1. *Suppose p satisfies complementarity, then for any two nonempty sets of agents B, $C \subset N$, we have $p(N) - p(N \backslash B) > p(N \backslash C) - p((N \backslash C) \backslash B)$.*

Proof. If p satisfies complementarity, then for two nonempty sets T and S with $T \subset S$ and two agents i, $j \notin S$, we have

$$\begin{aligned} p(S \cup \{i\} \cup \{j\}) - p(S) &= p(S \cup \{i\} \cup \{j\}) - p(S \cup \{i\}) + p(S \cup \{i\}) - p(S) \\ &> p(T \cup \{i\} \cup \{j\}) - p(T \cup \{i\}) + p(T \cup \{i\}) - p(T) \\ &= p(T \cup \{i\} \cup \{j\}) - p(T). \end{aligned}$$

This implies by induction that for any nonempty set $Q \subset N$ with $Q \cap S = \emptyset$ we have

$$p(S \cup Q) - p(S) > p(T \cup Q) - p(T). \tag{A.2}$$

Then, let $T = (N \backslash C) \backslash B$, $S = (N \backslash B)$, and $Q = B$. It is readily confirmed that $T \subset S$ and $Q \cap S = \emptyset$; thus, the lemma is proven using (A.2). □

From Lemma A.1, we conclude that $[p(N \backslash R_i) - p((N \backslash R_i) \backslash (\{i\} \cup M_i))]v_i^* < 1$. This means that i prefers to shirk whenever he sees someone shirking. Hence, s^* and the set of beliefs that are constructed above indeed constitute a PBE with full effort.

It remains to show that any alternative reward scheme v' with $v_i' < v_i^*$ cannot constitute a PBE with full effort. Suppose not, then the probability of success is $p(N)$ on the equilibrium path. If i shirks unilaterally, then he can at most trigger those in M_i to shirk, irrespective of the strategy profile. In other words, i's effort externality is confined to the coalition M_i. Since p is increasing, the difference in expected reward is less than the effort cost. Hence, i can make a profitable deviation by shirking, leading to a contradiction. Note that all these arguments go through for any fixed π, thus we have proven the proposition. □

Proof of Proposition 2.

Proof. As usual, we first prove that $\{\pi, v^*(\pi)\}$ is an EFI mechanism. Consider a strategy profile s^* with $s_i^* \equiv 1$, that is, an agent always exerts effort irrespective of his information set. This strategy profile can be sustained by a PBE with the set of beliefs that $a_j = 1$ for all $j \notin N_i$; that is, an agent believes that those whom he cannot see will exert effort. Note that if agent i sees no one shirking then he believes that all the other agents work. Hence, he prefers to work if and only if $[p(N) - p(N \backslash \{i\})]v_i \geq 1$, which holds for v_i^*. In contrast, if i sees a nonempty subset of agents $S_i \subseteq N_i$ who shirk, then his expected utility equals $p(N \backslash S_i)v_i^* - 1$ if he works; equals $p((N \backslash S_i) \backslash \{i\})v_i^*$ if he shirks. Then by substitutability, we have $p(N \backslash S_i) - p((N \backslash S_i) \backslash \{i\}) \geq p(N) - p(N \backslash \{i\})$. This implies that i still prefers to work. Hence, s^* and the set of beliefs constitute a PBE. Finally, we argue that there does not exist a reward scheme v' with $v_i' < v_i^*$ that admits a PBE with full effort. Suppose not, then i must prefer working to shirking if he encounters no shirking. Due to substitutability, if i shirks unilaterally then each $j \in M_i$ prefers to work, as argued above. This means that the difference in expected reward equals $p(N) - p(N \backslash \{i\})$. Since i is indifferent under v_i^*, he must prefer shirking under v', a contradiction. Hence, $v^*(\pi)$ is indeed optimal. □

Proof of Lemma 2.

Proof. Suppose not, then $\pi_i^* < \pi_{k'}^* < \pi_j^*$. Since (i,k'), $(j,k') \in g$, we have $j \in M_i^*$. Now switch i and j and call the new permutation π'. Since $\{k|(i,k) \in g\} = \{k|(j,k) \in g\}$, we have $(i,j) \notin g$, and thus, $N_i' = N_j^*$, $N_j' = N_i^*$, $M_i' = M_j^*$ and $M_j' \cup \{j\} = M_i^* \cup \{i\}$. Consider an agent $k \neq i$, j. There are three possibilities to consider. First, i, $j \notin M_k^*$. Since $N_i' = N_j^*$ and $N_j' = N_i^*$, the switch between i and j will not affect M_k, and thus, $M_k' = M_k^*$, meaning that $v_k^*(\pi') = v_k^*(\pi^*)$. Second, $i \in M_k^*$. It follows that $j \in M_k^*$ as $j \in M_i^*$. Since $M_i' = M_j^*$ and $M_j' \cup \{j\} = M_i^* \cup \{i\}$, the switch will not affect M_k; thus, $M_k' = M_k^*$ and $v_k^*(\pi') = v_k^*(\pi^*)$. Third, $j \in M_k^*$ but $i \notin M_k^*$. This means that $(i,k) \in g$, and by Lemma 1, that $\pi_i^* < \pi_k^* < \pi_j^*$. It follows that $M_k^* \backslash \{j\} = M_k' \backslash \{i\}$, and thus, we have

$$\begin{aligned} p(N \backslash (\{k\} \cup M_k^*)) &= p(N \backslash (\{k\} \cup (M_k^* \backslash \{j\}) \cup \{j\})) \\ &= p((N \backslash (\{k\} \cup (M_k^* \backslash \{j\}))\backslash \{j\}) \\ &= p((N \backslash (\{k\} \cup (M_k' \backslash \{i\}))\backslash \{j\}) \\ &> p((N \backslash (\{k\} \cup (M_k' \backslash \{i\}))\backslash \{i\}) \\ &= p(N \backslash (\{k\} \cup (M_k' \backslash \{i\}) \cup \{i\})) = p(N \backslash (\{k\} \cup M_k')). \end{aligned}$$

The inequality above is due to that i is more important than j. Then, from Proposition 1, we have $v_k^*(\pi') < v_k^*(\pi^*)$. Moreover, since $M_i' = M_j^*$, we have

$$p(N \backslash (\{j\} \cup M_j^*)) = p((N \backslash M_j^*)\backslash \{j\}) > p((N \backslash M_j^*)\backslash \{i\}) = p(N \backslash (\{i\} \cup M_i')).$$

It follows from Proposition 1 that $v_i^*(\pi') < v_j^*(\pi^*)$. Finally, since $M_j' \cup \{j\} = M_i^* \cup \{i\}$, we have $v_j^*(\pi') = v_i^*(\pi^*)$. This implies that the total payoffs to the agents are strictly lower under π' than under π^*, leading to a contradiction. Thus, the lemmas is proven. □

Proof of Lemma 3.

Proof. Define $\Delta MB(m) \equiv MB(m+1) - MB(m)$. From basic calculation, we have

$$\begin{aligned}\Delta MB(m) = &\sum_{i=1}^{n-3-m}\left[\frac{1}{p(N)-p(\{j|j<n-m-1\}\backslash\{i\})} - \frac{1}{p(N)-p(\{j|j<n-m-2\}\backslash\{i\})}\right]\\ &-\sum_{i=1}^{n-3-m}\left[\frac{1}{p(N)-p(\{j|j<n-m\}\backslash\{i\})} - \frac{1}{p(N)-p(\{j|j<n-m-1\}\backslash\{i\})}\right]\\ &+\left[\frac{1}{p(N)-p(\{j|j<n-m-1\})} - \frac{1}{p(N)-p(\{j|j<n-m\})}\right]\\ &+\left[\frac{1}{p(N)-p(\{j|j<n-m-2\})} - \frac{1}{p(N)-p(\{j|j<n-m\}\backslash\{n-m-2\})}\right].\end{aligned}$$

Note that given a specific i, the term

$$\left[\frac{1}{p(N)-p(\{j|j<n-m\}\backslash\{i\})} - \frac{1}{p(N)-p(\{j|j<n-m-1\}\backslash\{i\})}\right]$$

is decreasing in m. This is due to the exactly same reasoning in the proof of Corollary 1. Thus, the difference between the above two summations is negative. In addition, the value of the third and forth bracket in the expression of $\Delta MB(m)$ are both negative, since p is increasing. Therefore, $\Delta MB(m)$ is negative, meaning that $MB(m)$ is decreasing in m. □

Proof of Proposition 5.

Proof. We first prove that the marginal benefit is decreasing within each star. Consider a star i, with $1 \leq i \leq r$. Let m_i be the number of peripheries who act after the center within star i, with $0 \leq m_i \leq s_i - 1$. Given the result of Lemma 4, it is without loss of generality to assume that for any star $j < i$, $m_j = 0$, and that for any star $j > i$, $m_j = s_j - 1$. Note that the marginal benefit of raising m_i has two independent components: first, improving the transparency of the actions within i and reducing the number of the center's predecessors; second, improving the transparency of all preceding actions that can be learned by the center of star i on the equilibrium path. Lemma 3 has shown that the first component is decreasing in m_i, thus it suffices to show that the second component is also decreasing in m_i. Denote the second component $MB_{j<i}(m_i)$. Since $m_j = 0$ for any star $j < i$, the total payoffs to the agents of these stars, $v^*_{j<i}(m_i)$, is given by

$$v^*_{j<i}(m_i) = \sum_{j=1}^{i-1} \left[\underbrace{\frac{s_j - 1}{p(n) - p(\sum_{k=1}^{j-1} s_k + s_j + s_i - 3 - m_i)}}_{\text{payoffs to the peripheries}} + \underbrace{\frac{1}{p(n) - p(\sum_{k=1}^{j-1} s_k + s_j + s_i - 2 - m_i)}}_{\text{payoff to the center}} \right].$$

By definition, $MB_{j<i}(m_i) = v^*_{j<i}(m_i + 1) - v^*_{j<i}(m_i)$, with $0 \leq m_i \leq s_i - 2$. Since p satisfies complementarity, it can be easily shown that $MB_{j<i}(m_i)$ is indeed decreasing in m_i. Thus, the marginal benefit is decreasing within star i.

Then, we prove that the marginal benefit is decreasing across stars. This follows immediately from the proof of Lemma 4. Specifically, a periphery of star i who acts after the center of i can impose an implicit threat of shirking on more agents than his counterparts in any star $j < i$. This implies that the marginal benefit is decreasing across stars. In summary, the marginal benefit is decreasing through the process characterized by Proposition 5.

To see that the center of star 1 does not act the first, we consider the marginal benefit when $m_1 = s_1 - 2$ and $m_j = s_j - 1$ for all star $j > 1$. However, this is equal to the marginal benefit of a single star when $m = n - 2$, and from Proposition 4, $MB(n-2) < MC$. Thus, the center of star 1 does not act the first. Similarly, to see that the center of star r does not act the last, we consider the marginal benefit when $m_i = 0$ for any star i. However, this is equal to the marginal benefit of a single star when $m = 0$. From Proposition 4, we have

$$MB(0) = \frac{n-2}{p(n) - p(n-2)} - \frac{n-2}{p(n) - p(n-3)} + \frac{1}{p(n) - p(n-1)} > \frac{1}{p(n) - p(n-1)} = MC.$$

Thus, the center of star r does not act the last. Therefore, the proposition is proven. □

A.2 Optimal Sequence for Partially Adjustable Star

Here, we show that the algorithm of searching the optimal sequence for star networks remains valid if the relative order between the center and some peripheries is nonadjustable.

Suppose that due to technology constraint, a subset $K_1 \subset N$ of peripheries have to execute their tasks before the center, a subset $K_2 \subset N$ of peripheries have to execute their tasks after the center, and the remaining agents are perfectly flexible for ordering. Let t be the number of agents in the third group, and relabel the peripheries from 1 to $t - 1$, with a higher index referring to a more important agent. Let agent t be the center. Suppose that among these $t-1$ peripheries, m act after the center, then from Lemma 2, they are more important than the other peripheries. Thus, the total payoffs to the agents is given by

$$v^*(m) = \underbrace{\sum_{i=1}^{t-1-m} \frac{1}{p(N) - p(\{j|j < t-m\} \backslash \{i\} \cup K_1)}}_{\text{payoffs to the remaining predecessors}}$$

$$+ \underbrace{\frac{1}{p(N) - p(\{j|j < t-m\} \cup K_1)}}_{\text{payoff to the center}} + \underbrace{\sum_{t-m}^{t-1} \frac{1}{p(N) - p(N \backslash \{i\})}}_{\text{payoffs to the remaining successors}}$$

$$+ \underbrace{\sum_{k_1 \in K_1} \frac{1}{p(N) - p(\{j|j < t-m\} \cup K_1 \backslash \{k_1\})}}_{\text{payoff to the agents in } K_1} + \underbrace{\sum_{k_2 \in K_2} \frac{1}{p(N) - p(N \backslash \{k_2\})}}_{\text{payoffs to the agents in } K_2}.$$

Analogously, the marginal benefit $MB(m;t)$ is equal to

$$\sum_{i=1}^{t-2-m} \left[\frac{1}{p(N) - p(\{j|j < t-m\} \backslash \{i\} \cup K_1)} - \frac{1}{p(N) - p(\{j|j < t-m-1\} \backslash \{i\} \cup K_1)} \right]$$
$$+ \sum_{k_1 \in K_1} \left[\frac{1}{p(N) - p(\{j|j < t-m\} \cup K_1 \backslash \{k_1\})} - \frac{1}{p(N) - p(\{j|j < t-m-1\} \cup K_1 \backslash \{k_1\})} \right]$$
$$+ \frac{1}{p(N) - p(\{j|j < t-m\})},$$

and the marginal cost $MC(m;t)$ is equal to $[p(N) - p(N \backslash \{t-m-1\})]^{-1}$.

It can be proven analogously to Lemma 3 that $MB(m;t)$ is decreasing in m, as p satisfies complementarity. On the other hand, $MC(m;t)$ is increasing in m, as an agent with lower index is less important to the project. Thus, the optimizer m^* is either a corner solution or an interior solution such that $m^* = \min\{m | MB(m) \leq MC(m)\}$, meaning that the algorithm in Section 4 is still valid for searching the optimal sequence for star networks.

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