# Econ 201B TA Section: Week 1

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January 13, 2017

# 1 Practice Problems

See the solutions to Practice Problems.

# 2 Math Review

## 2.1 Separating Hyperplane Theorem

**Theorem 2.1.** *Suppose that the **convex** sets $A$ and $B \subset \mathbb{R}^N$ are disjoint (i.e., $A \cap B = \emptyset$). Then there is $p \in \mathbb{R}^N$ with $p \neq 0$, and a value $c \in \mathbb{R}$, such that $p \cdot x \geq c$ for every $x \in A$ and $p \cdot y \leq c$ for every $y \in B$. That is, there is a hyperplane that separates $A$ and $B$, leaving $A$ and $B$ on different sides of it.*

Let us use the Separating Hyperplane Theorem to prove the theorem in the following.

**Theorem 2.2.** *An undominated strategy $\sigma_i$ is a best response to some beliefs $\sigma_{-i} \in \Delta(S_{-i})$.*

*Proof.* It will be a bit more detailed than the lecture notes. Write $\overrightarrow{u}_i(\sigma_i) = (u_i(\sigma_i, s_{-i}))_{s_{-i}}$ for $i$'s utility vector under strategy $\sigma_i$. The set $\overrightarrow{U}_i$ of such vectors for all $\sigma_i$ is convex, since $\overrightarrow{u}_i(\sigma_i)$ is linear in $\sigma_i$. Now consider an undominated strategy $\tilde{\sigma}_i$. Write $\overrightarrow{U}^u_i(\tilde{\sigma}_i)$ for the set of utility vectors that dominate $\overrightarrow{u}_i(\tilde{\sigma}_i)$ (i.e., the strict upper contour set). Note that $\overrightarrow{U}^u_i(\tilde{\sigma}_i)$ is also convex. Hence, from the Separating Hyperplane Theorem, there is a hyperplane with a normal vector $\sigma_{-i}$, such that $\sigma_{-i} \cdot \overrightarrow{u}^u_i \geq \sigma_{-i} \cdot \overrightarrow{u}_i(\tilde{\sigma}_i)$ for all $\overrightarrow{u}^u_i \in \overrightarrow{U}^u_i(\tilde{\sigma}_i)$. It can be easily shown that $\sigma_{-i}$ is non-negative. Suppose not, then there is an $s_{-i}$ such that $\sigma_{-i}(s_{-i}) < 0$. Pick a $\overrightarrow{u}^u_i \in \overrightarrow{U}^u_i(\tilde{\sigma}_i)$, such that $\overrightarrow{u}^u_i(s_{-i}) - \overrightarrow{u}_i(\tilde{\sigma}_i, s_{-i}) = k$ for some $k > 0$, and $\overrightarrow{u}^u_i(s'_{-i}) - \overrightarrow{u}_i(\tilde{\sigma}_i, s'_{-i}) < \varepsilon$ for all $s'_{-i} \neq s_{-i}$ and some $\varepsilon > 0$. Fix $k$, for sufficiently small $\varepsilon$, we have

$$\sigma_{-i} \cdot (\overrightarrow{u}_i^u - \overrightarrow{u}_i(\tilde{\sigma}_i)) = \sigma_{-i}(s_{-i})k + \sum_{s'_{-i} \neq s_{-i}} \sigma_{-i}(s'_{-i})\varepsilon < 0.$$

This means that $\sigma_{-i} \cdot \overrightarrow{u}_i^u < \sigma_{-i} \cdot \overrightarrow{u}_i(\tilde{\sigma}_i)$, leading to a contradiction, so $\sigma_{-i}$ is non-negative. Pick an sequence of utility vectors $\{\overrightarrow{u}_n^u\}$, which converge to $\overrightarrow{u}_i(\tilde{\sigma}_i)$, from the set $\overrightarrow{U}_i^u(\tilde{\sigma}_i)$. From the Separating Hyperplane Theorem, for each $\overrightarrow{u}_n^u$, we have

$$\sigma_{-i} \cdot \overrightarrow{u}_n^u \geq \sigma_{-i} \cdot \overrightarrow{u}_i(\sigma_i), \text{ for all } \sigma_i$$

Taking limit, we have $\sigma_{-i} \cdot \overrightarrow{u}_i(\tilde{\sigma}_i) \geq \sigma_{-i} \cdot \overrightarrow{u}_i(\sigma_i)$ for all $\sigma_i$, i.e.,

$$u_i(\tilde{\sigma}_i, \sigma_{-i}) = \sum_{s_{-i}} \sigma_{-i}(s_{-i})u_i(\tilde{\sigma}_i, s_{-i}) \geq \sum_{s_{-i}} \sigma_{-i}(s_{-i})u_i(\sigma_i, s_{-i}) = u_i(\sigma_i, \sigma_{-i}),$$

that is, $\tilde{\sigma}_i$ is a best response to $\sigma_{-i}$. □

## 2.2 Supporting Hyperplane Theorem

**Theorem 2.3.** *Suppose that $B \subset \mathbb{R}^N$ is* ***convex*** *and that $x$ is not an element of the interior of set $B$ (i.e., $x \notin Int\ B$). Then there is $p \in \mathbb{R}^N$ with $p \neq 0$ such that $p \cdot x \geq p \cdot y$ for every $y \in B$.*

Try to prove Theorem 2.2 by Theorem 2.3. It is indeed easier than the previous practice.

## 2.3 Kakutani's Fixed Point Theorem

**Definition 2.1 (upper hemicontinuity).** Given $A \subset \mathbb{R}^N$ and the closed set $Y \subset \mathbb{R}^K$, the correspondence $f : A \rightrightarrows Y$ is upper hemicontinuous (uhc) if it has a closed graph and the images of compact sets are bounded, that is, for every compact set $B \subset A$ the set $f(B) = \{y \in Y : y \in f(x) \text{ for some } x \in B\}$ is bounded.

**Theorem 2.4.** *Suppose that $A \subset \mathbb{R}^N$ is a nonempty, compact, convex set, and that $f : A \rightrightarrows A$ is an upper hemicontinuous correspondence from $A$ into itself with the property that the set $f(x) \subset A$ is nonempty and convex for every $x \in A$. Then $f(x)$ has fixed point; that is, there is an $x \in A$ such that $x \in f(x)$.*

We will use Kakutani's Fixed Point Theorem to prove the existence of Nash equilibrium in the subsequent lectures.

**Econ 106G Spring 2015**
**Practice Problems 1: Basic Concepts, Dominated Stratgies**

**You do not need to submit answers to these problems. I still encourage you to think them through thoroughly, write down your answers and discuss them with your classmates.**

1. **Basic Concepts:** For each of the games introduced in the first lecture, determine

   (a) whether we represented the game as an extensive-form game or as a normal-form game

   (b) whether or not the game is of complete information or incomplete information

2. **Players, strategies and utilities:** For each of the games, determine

   (a) who are the players

   (b) what are their actions and strategies

   (c) what are their payoffs (they were not always specified, so you will sometimes have to use your judgment)

3. **Another game:** In analogy to the games presented in class, find a strategic situation from current events or your own experience - the strategic interaction between an economics professor and his class, if you wish - and model it as a game. State explicitly, who are the players, and what are their actions, strategies, and utilities in your game. If the game is an extensive-form game, draw the game tree. If it is a normal form game, write down the payoff matrix.

4. **A date:** Jane and John have loosely planned to meet for an ice cream sandwich at Diddy Riese's icecream place at 7pm. At 6.45pm each of them considers individually, whether to carry through with the plan or to stay at home and watch television. They would both like to meet each other at the ice cream place and would receive utility 1 from doing so. If one of them does not show up, the other one is just indifferent between ice cream and TV. They attach utility 0 to these outcomes.

| | | John: Ice cream | John: TV |
|---|---|---|---|
| Jane | Ice cream | 1 , 1 | 0 , 0 |
| | TV | 0 , 0 | 0 , 0 |

   (a) If you were Jane's friend, would you recommend her to go for ice cream or stay at home and watch TV?

   (b) Is your recommendation a strictly dominant strategy?

   (c) If Jane goes for ice cream, what are John's best responses? What if Jane chooses TV?

# Solutions to Practice Problems

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## Question 1 and 2

### A Beauty Contest

The game should be represented as a normal-form game and is of complete information.

- *Players: student* $i \in \{1, 2, \dots, N\} = I$, $N \geq 2$.
- *Strategies:* $s_i \in \{1, 2, \dots, 100\} = S$.
- *Payoff: for each* $s = (s_i, s_{-i})$,

$$u_i(s_i, s_{-i}) = \begin{cases} V_{OR} & \textit{if } s_i = argmin_{\{s_j\}_{j \in I}} \; |s_j - \frac{2}{3} \cdot \frac{1}{N} \sum_{j=1}^{N} s_j| \\ 0 & \textit{otherwise} \end{cases}$$

### The Penny Auction

The game should be represented as an extensive-form game and is of complete information.

- *Players: student* $i \in \{1, 2, \dots, N\} = I$, $N \geq 2$.
- *Strategies:* $s_i = \left\{ b_n(H_n) = 1 \textit{ or } 0, \; \forall \; H_n \in \mathcal{H}_n \right\}_{n=1}^{\infty}$, where $\mathcal{H}_n$ denotes the set of all bidding histories by the round $n$.
- *Payoff: for each complete bidding history* $H \in \mathcal{H}$,

$$u_i(H) = \begin{cases} V_{FT} - c \cdot \sum_{n=1}^{\infty} \mathbb{1}\{b_n(H_n) = 1\} & \textit{if it is the last bid} \\ -c \cdot \sum_{n=1}^{\infty} \mathbb{1}\{b_n(H_n) = 1\} & \textit{otherwise} \end{cases}$$

## Crying it out

The game should be represented as an extensive-form game and is of incomplete information.

- *Players: Daughter (D) and Father (F); Daughter wants the doll either very much (H) or not so much (L).*
- *Strategies: Draw the game tree.* $s_D = \{reasonable \text{ or } stubborn\}_{\mathscr{I}_D}$, *where* $\mathscr{I}_D$ *denotes the set of information sets at which D makes a move;* $s_F = \{push\ over \text{ or } strict\}_{\mathscr{I}_F}$, *where* $\mathscr{I}_F$ *denotes the set of information sets at which F makes a move.*
- *Payoffs: let* $n$ *be the number of total rounds*

$$u_D^H(n) = \begin{cases} V_D^H - nc_D & \text{if } F \text{ gives in} \\ -nc_D & \text{if } D \text{ gives in} \end{cases} \qquad u_D^L(n) = \begin{cases} V_D^L - nc_D & \text{if } F \text{ gives in} \\ -nc_D & \text{if } D \text{ gives in} \end{cases}$$

$$u_F(n) = \begin{cases} (1-\mu_n^f)V_F^L + \mu_n^f V_F^H - nc_F & \text{if } F \text{ gives in} \\ (1-\mu_n^d)V_F^L + \mu_n^d V_F^H - nc_F & \text{if } D \text{ gives in} \end{cases}$$

where $\mu_n^f$ (resp. $\mu_n^d$) is $F$'s belief of $D$ being the High type given that $F$ (resp. $D$) gives in in round $n$.

## The Sex-Ratio

The game should be represented as a normal-form game and is of complete information.

- *Players: Women in the population.*
- *Strategies:* $s_i \in \{1/2, 1/6\} = S$, *representing the proportion of female offspring.*
- *Payoff: The number of grandchildren.*

| | | The Rest of Population: 1/2 | The Rest of Population: 1/6 |
|---|---|---|---|
| Individual | 1/2 | 144 | 86.4 |
| | 1/6 | 144 | 48 |

## Question 3

Try to model a game on your own.

## Question 4

**(a)** if John goes to the ice cream, Jane should go to the ice cream as well, otherwise Jane will be indifferent between ice cream and TV. Thus, I would recommend her to go for ice cream.

**(b)** No, it's not strictly dominant (only weakly dominant).

**(c)** If Jane goes for ice cream, the UNIQUE best response of John is to go for ice cream. But if Jane chooses TV, then John has two best responses: "ice cream" and "TV", since he is indifferent.

# Econ 201B TA Section: Week 2

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## 1 Review of Concepts

We summarize the relationships between different kinds of strategy that we have discussed.

$$\{\textit{dominant strategy}\} \subseteq \{\textit{weakly dominant strategy}\} \subseteq \{\textit{rationalizable strategy}\}$$
$$\subseteq \{\textit{best response strategy}\} = \{\textit{level-1 rationalizable strategy}\} = \{\textit{undominated strategy}\}$$

## 2 Elimination of Weakly Dominated Strategies

The following example illustrates the difficulties that may occur when eliminating weakly dominated strategies.

| | | Player 2 | |
|---|---|---|---|
| | | L | R |
| | U | 1, 1 | 0, 0 |
| Player 1 | M | 1, 1 | 2, 1 |
| | D | 0, 0 | 2, 1 |

The actions that survive iterative elimination of weakly dominated strategies can depend on the order in which the actions are eliminated. For example, $U$ can be eliminated since it is weakly dominated by $M$, and then $L$ can be eliminated since it is weakly dominated by $R$. Now, Player 2 will choose action $R$, which will result in a payoff of $(2, 1)$ for which ever action Player 1 selects. On the other hand, action $D$ could have been eliminated first since it is weakly dominated by $M$, and then R could have been eliminated since it is weakly dominated by $L$. Now, the payoff is $(1, 1)$ for which ever action Player 1 selects.

# 3 Electoral Competition with 3 Players

Consider a variation of the electoral competition in lecture.

- Players: $i = A, B, C$. Three parties
- Strategy Sets: $S_1 = S_2 = S_3 = \{1, 2, 3, \ldots, 9\}$.
- Payoffs: for each policy, $1, 2, \ldots, 9$, there are 100 voters to support it. Each voter votes for the party whose policy is closest to the policy he or she supports. If multiple parties are equally attractive to a group of voters, then the group is split equally.

Is this game dominance solvable?

## Solution:

This game is not dominance solvable. It suffices to prove that there is no dominated strategy. Due to the symmetry, we only need to focus on $\{1, 2, 3, 4, 5\}$. Consider Player A's problem. Without loss of generality, assume that $s_B \leq s_C$. It is easy to see that $s_A = 1$ is undominated by $s'_A = 2$, as $u_A(1, 2, 3) = u_A(2, 2, 3) = 100$. Also, $s_A = 1$ is undominated by $s'_A = 9$, as $u_A(1, 7, 8) = 350 > u_A(9, 7, 8) = 100$. For any $s'_A \in [3, 8]$, one can always make $s_B = s'_A - 1$ and $s_C = s'_A + 1$ such that $u_A(s'_A, s_{-A}) = 100$, while $u_A(1, s_{-A}) = \frac{s_B}{2} \times 100 = 50(s'_A - 1)$, which is at least 100. Thus, $s_A = 1$ is undominated. Moreover, we have learned that for any $s'_A \in [2, 8]$, there exists $s_{-A}$ with $s_B = s'_A - 1$ and $s_C = s'_A + 1$ such that $u_A(s'_A, s_{-A}) = 100$.

Now, consider $s_A = 2$. It is undominated by $s'_A = 1$, since $u_A(2, 2, 3) = u_A(1, 2, 3) = 100$. It is also undominated by $s'_A = 9$, since $u_A(2, 7, 8) = 400 > u_A(9, 7, 8) = 100$. Then, for any $s'_A \in [3, 8]$ with $s_B = s'_A - 1$ and $s_C = s'_A + 1$, we have $u_A(s'_A, s_{-A}) = 100$. However, for every such $s_{-A}$, $u_A(2, s_{-A})$ is at least 100. Thus, $s_A = 2$ is undominated.

Consider $s_A = 3$. It is undominated by $s'_A = 1$, as $u_A(3, 2, 3) = 350 > u_A(1, 2, 3) = 100$. It is also undominated by $s'_A = 9$, since $u_A(3, 7, 8) = 450 > u_A(9, 7, 8) = 100$. Then, for any $s'_A \in \{2\} \cup [4, 8]$ with $s_B = s'_A - 1$ and $s_C = s'_A + 1$, we have $u_A(s'_A, s_{-A}) = 100$, while for every such $s_{-A}$, $u_A(3, s_{-A})$ is at least 150. Thus, $s_A = 3$ is undominated.

Consider $s_A = 4$. It is undominated by $s'_A = 1$, as $u_A(4, 2, 3) = 600 > u_A(1, 2, 3) = 100$. It is also undominated by $s'_A = 9$, since $u_A(4, 7, 8) = 500 > u_A(9, 7, 8) = 100$. Then, for any $s'_A \in [2, 3] \cup [4, 8]$ with $s_B = s'_A - 1$ and $s_C = s'_A + 1$, we have $u_A(s'_A, s_{-A}) = 100$, while for every such $s_{-A}$, $u_A(4, s_{-A})$ is at least 200. Thus, $s_A = 4$ is undominated.

Finally, $s_A = 5$ is undominated by $s'_A = 1$, because $u_A(5, 2, 3) = 550 > u_A(1, 2, 3) = 100$. It is also undominated by $s'_A = 9$, since $u_A(5, 7, 8) = 550 > u_A(9, 7, 8) = 100$. Then, for any

$s'_A \in [2,4] \cup [6,8]$ with $s_B = s'_A - 1$ and $s_C = s'_A + 1$, we have $u_A(s'_A, s_{-A}) = 100$, while for every such $s_{-A}$, $u_A(5, s_{-A})$ is at least 250. Thus, $s_A = 5$ is undominated.

In summary, there is no dominated strategy, hence the game is not dominance solvable.

# 4 Joint Project

Two people are engaged in a joint project. Each person $i$ puts in the effort $x_i \in [0,1]$, which costs her $c(x_i)$; the outcome of the project is worth $f(x_1, x_2)$. The value of the project is split equally between the two people, regardless of their effort levels.

(a) Formulate this situation as a game.

(b) If $f(x_1, x_2) = 3x_1x_2$; $c(x_i) = x_i^2$, $i = 1, 2$, then is this game dominance solvable?

(c) If $f(x_1, x_2) = 4x_1x_2$; $c(x_i) = x_i$, $i = 1, 2$, then is this game dominance solvable?

## Solution:

**(a)** The game is characterized as follows.

- Players: $i = 1, 2$.
- Strategies: $x_i \in [0,1]$.
- Payoffs: $u_i(x_i, x_j) = f(x_1, x_2)/2 - c(x_i)$.

**(b)** For each $x_j$, Player $i$'s best response $x_i^*$ is the solution to the following problem.

$$\max_{x_i} \frac{3x_ix_j}{2} - x_i^2$$

From FOC, we have

$$x_i^* = \frac{3x_j}{4}, \; i = 1, 2$$

Due to the strict quasiconcavity of the utility function and that $x_i \in [0,1]$, $i = 1, 2$, we have that any $x_i \in (\frac{3}{4}, 1]$ is dominated by $x_i = \frac{3}{4}$. Eliminating iteratively dominated strategies, we find that the unique outcome is $(x_1, x_2) = (0, 0)$. Thus, the game is dominance solvable. Figure 1 illustrates how players' best response converge to $(0, 0)$ starting from an arbitrary point.

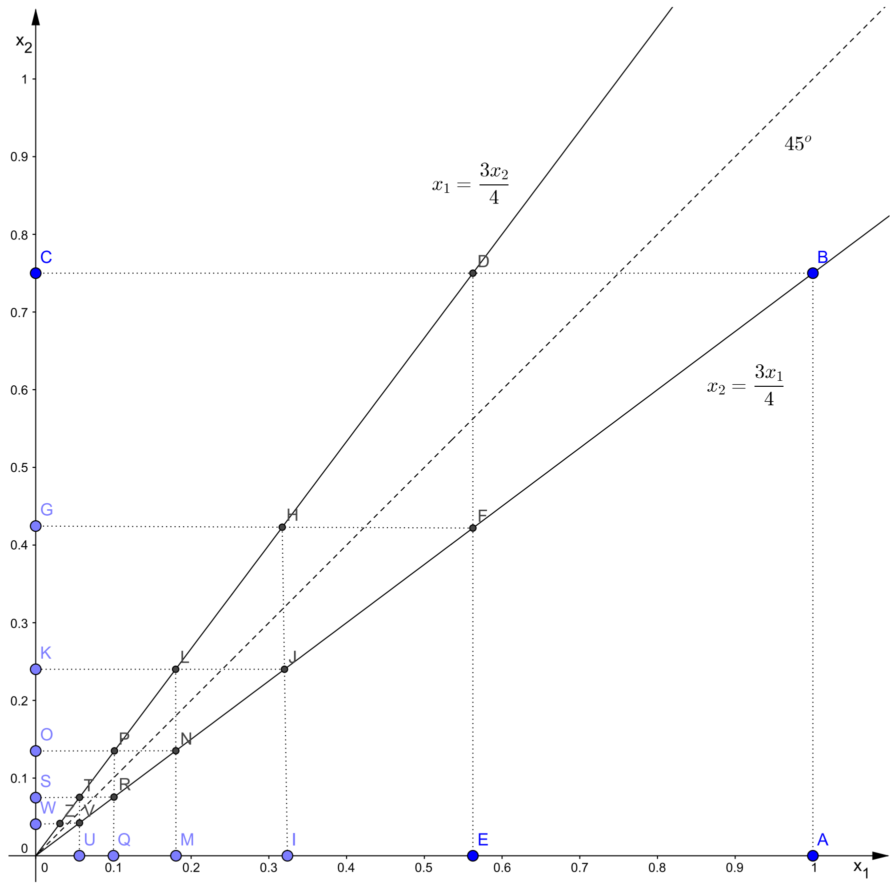


Figure 1: **Iterative Elimination of Dominated Strategy.**

We can see that this outcome is inefficient, which yields a zero social surplus. The first best outcome is $(1, 1)$ which yields a social surplus equal to 1. We call this situation "the tragedy of the commons".

**(c)** For each $x_j$, Player $i$'s best response $x_i^*$ is the solution to the following problem.

$$\max_{x_i} (2x_j - 1)x_i$$

Obviously, if $x_j > 1/2$, then $x_i^* = 1$; otherwise, $x_i^* = 0$. The best responses for each player is illustrated in Figure 2. For each player, there is no dominated strategy, thus the game is not dominance solvable. The game has two stable outcomes, namely, $(x_1, x_2) = (0, 0)$ and $(x_1, x_2) = (1, 1)$. The first is Pareto dominated by the second; the second is also the first best outcome.

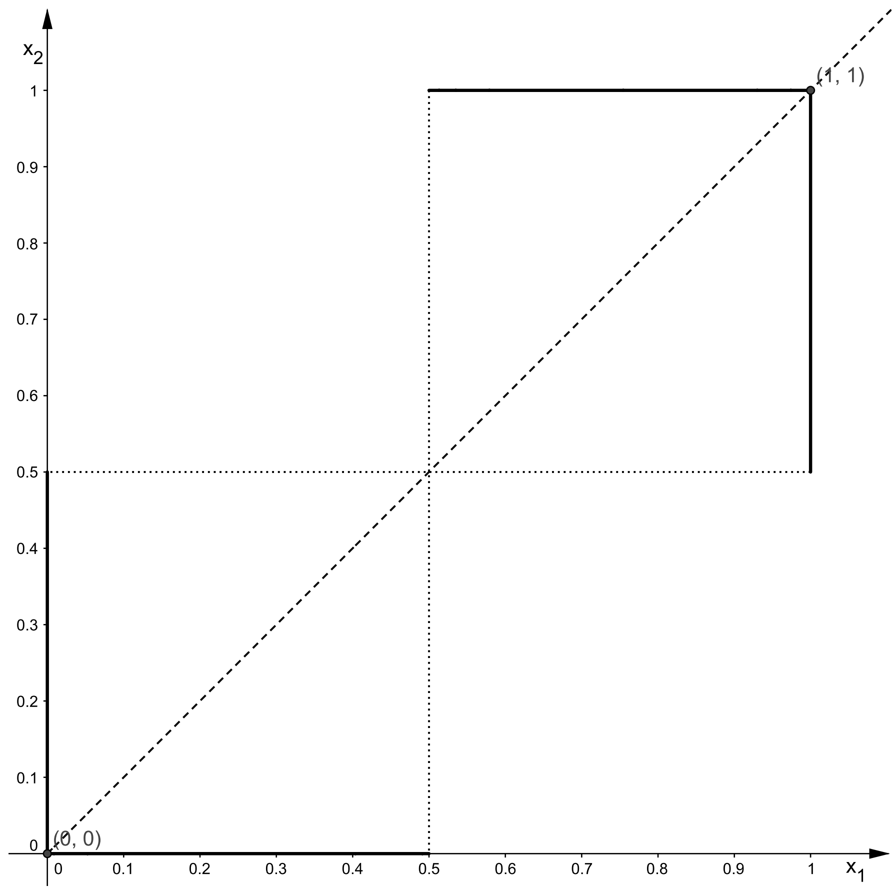


Figure 2: **Best Responses.**

# Econ 201B TA Section: Week 3

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## 1 Difference Between Mixed and Pure Strategy NE

Unlike in a pure strategy Nash equilibrium, in a mixed strategy equilibrium players are only responding optimally to others' mixed strategy, but in general not to the realized actions.
**Examples.**

- Matching pennies
- "Rock, Paper, Scissors"

## 2 Horizontally Differentiated Bertrand Competition

Two firms 1, 2 are selling imperfectly substitutable goods. They are competing in Bertrand fashion by posting prices $p_i \in [0, 10]$. Demand for firm $i$'s good is $q_i(p_i, p_{-i}) = 4 - p_i + p_{-i}$ (the demand increases in the competitor's price $p_{-i}$ because some consumers shift to firm $i$ when $p_i$ goes up). There are no production costs. Thus the profit functions of the firms are

$$\pi_i(p_i, p_{-i}) = p_i(4 - p_i + p_{-i})$$

The FOC and concavity yields the best response functions

$$p_i^*(p_{-i}) = \frac{4 + p_{-i}}{2}.$$

In contrast to Cournot competition, the best-response functions are increasing. We thus speak of strategic complements (See Figure 1). Clearly, there is a unique Nash equilibrium. It is unique with $p_1 = p_2 = 4$.

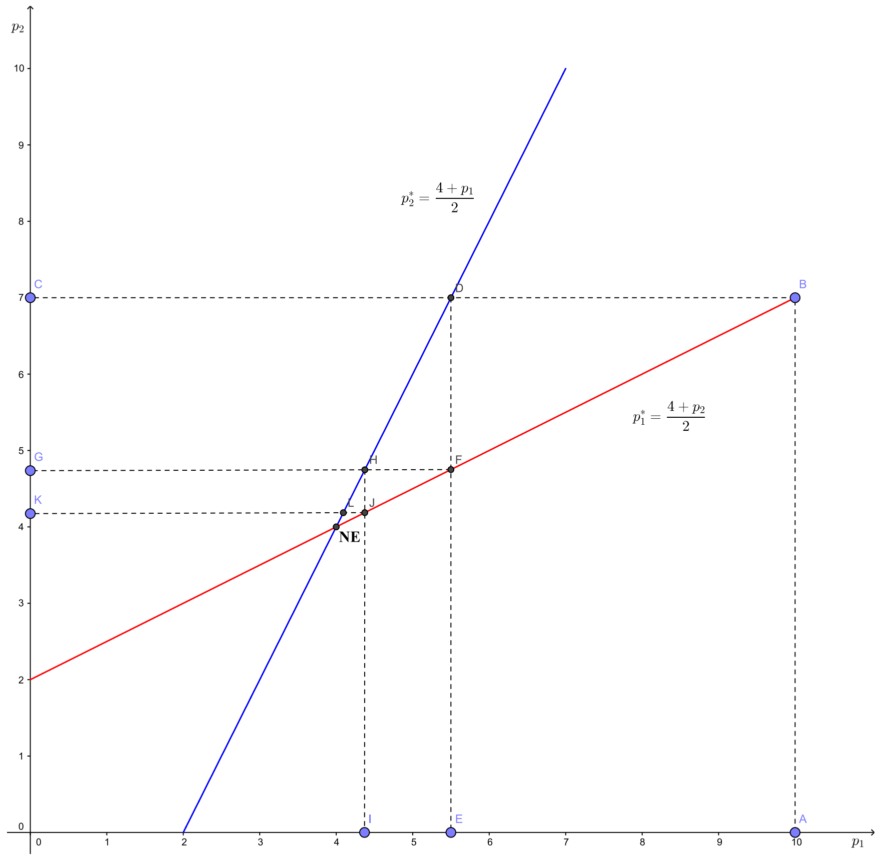


Figure 1: **Best Responses and Nash Equilibrium.**

# 3 Practice Problems 2

See the solutions posted online.

# Econ 201B TA Section: Week 4

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# 1 Quantal Response

## 1.1 Quantal Response Equilibrium

Quantal response equilibrium (QRE) is a solution concept in game theory. It provides an equilibrium notion with bounded rationality. QRE is not an equilibrium refinement, and it can give significantly different results from Nash equilibrium. QRE is only defined for games with discrete strategies, although there are continuous-strategy analogues.

**Definition 1.1.** A strategy profile $\sigma^*$ is a QRE if Player $i$ plays strategy $s_i$ with probability $\sigma_i^*(s_i)$ such that

$$\sigma_i^*(s_i) = \frac{\exp(u_i(s_i, \sigma_{-i}^*)/\mu)}{\sum_{s_i'} \exp(u_i(s_i', \sigma_{-i}^*)/\mu)} \tag{**}$$

For fixed $\mu \geq 0$, looking for a QRE is equivalent to looking for a fixed point. As $\mu \to 0$, QRE converges to NE. However, the theory is silent about where $\mu$ comes from.

**Example.** Traveler's dilemma. `https://en.wikipedia.org/wiki/Traveler%27s_dilemma`

## 1.2 Quantal Response Rationalizability

QRE is not a normative theory since it forces players to make mistakes. But the quantal-response idea more broadly also has normative appeal, when applied to the uncertainty of others' strategies:

- I best-respond to my belief $\sigma_{-i}^1$. That is, my probabilities of playing $s_i$ is given by $(**)$ with $\sigma_{-i}^* = \sigma_{-i}$ and $\mu = \mu^1 = 0$.

- My beliefs about others' strategies $\sigma^1_{-i}$ in turn are restricted by their rationality. But, I'm not so certain about that rationality. Specifically, I believe that opponent $j$ forms some beliefs about opponent's strategies $\sigma^2_{-j}$, and the frequency with which he chooses strategy $s_j$ is determined by

$$\sigma^1_j(s_j) = \frac{\exp(u_j(s_j, \sigma^2_{-j})/\mu^2)}{\sum_{s'_j} \exp(u_j(s'_j, \sigma^2_{-j})/\mu^2)}$$

  where $\mu_2 > 0$, reflecting the fact that I'm not sure about $j$'s rationality.

- In turn, my beliefs about $j$'s beliefs about other strategies $\sigma^2_j$, and all higher-order beliefs are determined similarly via

$$\sigma^N_j(s_j) = \frac{\exp(u_j(s_j, \sigma^{N+1}_{-j})/\mu^{N+1})}{\sum_{s'_j} \exp(u_j(s'_j, \sigma^{N+1}_{-j})/\mu^{N+1})}$$

If at some level of rationality $N$, I loose all faith, $\mu^N = \infty$, then $\sigma^{N-1}$ is uniform (and in particular independent of $\sigma^N$), and we can recursively determine the conjectures $\sigma^n$ at all levels $n \in \{1, \ldots, N-1\}$.

# 2 Correlated Equilibrium

## 2.1 Practice Problem 1

Consider the following game in normal form, where Player 1 chooses rows, Player 2 chooses columns, and player 1's payoff is listed first.

| | a | b | c |
|---|---|---|---|
| A | 5, 3 | 5, 5 | 3, 4 |
| B | 4, 10 | 9, 9 | 4, 11 |
| C | 1, 1 | 11, 4 | 2, 2 |

Completely characterize the set of correlated equilibria for this game. Are there any correlated equilibria that are not Nash equilibria?

**Solution.** Only $\{B, C\}$ and $\{b, c\}$ survive IESDS. Suppose $\sigma^*$ is a correlated equilibrium, then by definition we have

$$\sigma^*(B,b)(9-11)+\sigma^*(B,c)(4-2)\geq 0$$
$$\sigma^*(C,b)(11-9)+\sigma^*(C,c)(2-4)\geq 0$$

Since the payoff is symmetric, $\sigma^*$ is a correlated equilibrium if an only if

$$\sigma^*(B,c)\geq\sigma^*(B,b)$$
$$\sigma^*(C,c)\geq\sigma^*(C,b)$$

subject to $\sigma^*(B,c)+\sigma^*(B,b)+\sigma^*(C,c)+\sigma^*(C,b)=1$. Thus, there are a continuum of correlated equilibria.

Note that there are three Nash equilibria: $\{B,c\}$, $\{c,B\}$ and $\{\frac{1}{2}B+\frac{1}{2}C,\frac{1}{2}b+\frac{1}{2}c\}$. Thus, there are a continuum of correlated equilibria that are not Nash equilibrium.

## 2.2 Practice Problem 2

Can there be a correlated equilibrium where every player gets less than her lowest Nash equilibrium payoff? Explain or give an example.

**Solution.** Yes. Consider the following game between Player 1 and Player 2

| | a | b | c |
|---|---|---|---|
| A | 5, 2 | 0, 0 | 1, 1 |
| B | 6, 0 | 9, 9 | 0, 0 |
| C | 4, 4 | 0, 6 | 2, 5 |

There are two Nash equilibria: $\{B,b\}$ and $\{\frac{1}{2}A+\frac{1}{2}C,\frac{1}{2}a+\frac{1}{2}c\}$. The associated payoffs are $(9,9)$ and $(3,3)$, respectively.

Consider the following randomize device

| | a | b | c |
|---|---|---|---|
| A | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ |
| B | 0 | 0 | 0 |
| C | 0 | 0 | $\frac{1}{3}$ |

We can show that this gives us a correlated equilibrium $\sigma^*$ with both players get 8/3, which is less than they get when they play any Nash equilibrium.

# Econ 201B TA Section: Week 5

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## 1 Nim 2

Suppose there are $N$ matches and 2 players who take turns to pick matches. Both players can pick 1, 2, or 3 matches when it is their turn. The player who picks the last match loses this game. Let's assume that player 1 moves first. For what value of $N$ will player 2 have a winning strategy?

**Solution.** We analyze this problem by induction. If $N = 1$, player 1 definitely loses. If $N = 2$, player 1 picks 1 match and player 2 loses. If $N = 3$, player 1 picks 2 match and player 2 loses. If $N = 4$, player 1 picks 3 match and player 2 loses. If $N = 5$, regardless of how much player 1 picks, player 2 can always pick the 4th match and win. We can now summarize the pattern that for $N = 4k + 1$, $k = 0, 1, 2, \cdots$, player 2 has a winning strategy. Formally, we already show that this is true for $k = 0$ and $k = 1$. Suppose this conjecture is true for $N = 4m + 1$, meaning that player 2 can guarantee herself to pick the $4m$-th match. Then, for $N = 4(m + 1) + 1$, by picking the $4m$-th match, player 2 can leave 5 matches to player 1. As we have shown, player 2 wins for $N = 5$, and thus player 2 wins finally. This proves the conjecture. Moreover, when $N = 4k + 1$, player 2's winning strategy is always picking the $4k$-th match.

On the other hand, if $N \neq 4k + 1$, then player 1 has a winning strategy. Specifically, if $N = 4k$, then player 1 should pick the $(4k - 1)$-th match; if $N = 4k + 2$, then player 1 should pick the $(4k - 1)$-th match; if $N = 4k + 2$, then player 1 should pick the $(4k + 1)$-th match; if $N = 4k + 3$, then player 1 should pick the $(4k + 2)$-th match.

## 2 Nim 3

There are three piles of matches, with $n_1 = 3$, $n_2 = 4$, $n_3 = 5$ matches. When it is her turn each player, 1 or 2, chooses one pile and takes as many matches from this pile as she wants, but at least one. The player who picks the last match wins the game. One can use backward induction to show that player 1 has a winning strategy.

**Solution.** Using backward induction. Suppose the game ends in round $T$ in which player 1 picks the last match. Player 1 can guarantee this if at the beginning of round $T-1$, there are two piles left and each has one match. This can be guaranteed if at the beginning of round $T-2$ there are two piles left and one pile has one match. This can be guaranteed if at the beginning of round $T-3$ there are three piles left and one pile has one match. This can be guaranteed if at the beginning of round $T-4$, which is player 1's turn, there are three piles left. Therefore, player 1 has a winning strategy, which is just the reverse of the previous process.

## 3 Practice Problems 3

See the solutions.