Competitive Nonlinear Pricing for Signals

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September 1, 2018

Abstract

This paper studies nonlinear pricing for horizontally differentiated products that provide signaling value to consumers, who choose how much to purchase as a signal to the receivers. We characterize the optimal symmetric price schedules under different market structures. Under monopoly, when the receivers observe the price schedule, the market is partially covered, and quantity is downward distorted if there is slight horizontal differentiation. As the degree of horizontal differentiation rises, the market coverage rises, and the downward distortion decreases. When the degree is sufficiently high, for a certain level of signaling intensity, the monopolistic allocation achieves the first-best; for higher signaling intensities, quantity is upward distorted at the low end. In contrast, when the receivers do not observe the price schedule, the market is always partially covered, and the allocation is more dispersed than that in the observed case. Specifically, higher types purchase more than in the observed case, with the highest types purchasing more than the first-best, whereas lower types purchase less than in the observed case, with more types excluded from the market. When the market structure changes from monopoly to duopoly, market competition results in a higher market coverage and larger quantities for both the observed and unobserved case.

Keywords: Nonlinear pricing, Signaling, Signal jamming, Product differentiation, Market coverage, Quantity distortion

JEL Classification: D40, D82, L10

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*School of Management, Fudan University. Email: luzhuoran@fudan.edu.cn. I am greatly indebted to my advisors, Simon Board and Moritz Meyer-ter-Vehn, for their guidance and encouragement. I am grateful to Tomasz Sadzik for many highly valuable discussions. I have received many helpful comments from Justin Johnson, Jay Lu, Ichiro Obara, John Riley and Michael Waldman. I also wish to thank the seminar audiences at UCLA, ES North American Summer Meeting 2017, Tsinghua SEM, Cornell Johnson College of Business and Fudan School of Management. All remaining errors are mine.

1 Introduction

Starting with the seminal work of Mussa and Rosen (1978) and Maskin and Riley (1984) on monopolistic nonlinear pricing, there is a large literature on nonlinear pricing in competitive settings. These models typically assume that buyers derive intrinsic value from consuming the products. Recently, Lu (2018) studies monopolistic nonlinear pricing for products that provide signaling values to consumers and assesses how the transparency of pricing affects the degree of signaling and welfare. In contrast, this paper studies nonlinear pricing for horizontally differentiated products that provide signaling value to consumers, and further investigates how (horizontal) competition affects sellers' pricing strategies and the degree of signaling and welfare. The paper is also closely related to Rochet and Stole (2002) and Yang and Ye (2008) in the sense that the only substantial difference is that the products in our model have signaling value in addition to intrinsic value. Thus, our paper is complementary to the three recent papers, and establishes a close connection between each other.

In this paper, we derive the optimal symmetric price schedules, under different market structures, for horizontally differentiated products that provide signaling value to consumers with private information. The equilibrium depends critically on whether the signal receivers observe the sellers' price schedules, as well as on the market structure. We first consider the case in which a monopolist maximizes the joint profit of all products. When the receivers observe each product's price schedule, the (vertical) market is partially covered, and quantity is downward distorted if there is slight horizontal differentiation. As consumers' valuations for the product become more horizontally differentiated, the market coverage rises, and the downward distortion decreases. When the degree of horizontal differentiation is sufficiently high, for some intermediate level of signaling intensity, the monopolistic allocation can in fact achieve the first-best; for higher signaling intensities, quantity is upward distorted at the low end. In contrast, when the receivers do not observe any product's price schedule, the market is always partially covered, and the allocation is more dispersed than in the observed case. Specifically, an interval of higher types purchase more than in the observed case, with the highest types purchasing more than the first-best, whereas the rest types purchase less than in the observed case, with more types excluded from the market. When the market structure changes from monopoly to duopoly, in which each seller maximizes the profit of own product, market competition results in a higher market coverage and larger quantities for both the observed and unobserved case.

For the purpose of exposition, we present our model in terms of Spence's education model (Spence 1973) with productive education. In the model, two identical schools choose their own tuition scheme, and a worker chooses which school to attend and how much education to purchase to signal his privately know ability (*vertical type*) to competing employers. The worker's ability distributes uniformly over $[0, 1]$. Following Yang and Ye (2008), we model horizontal differentiation by assuming that the worker incurs transportation costs to attend school. The worker's distance to a school (*horizontal type*) distributes uniformly over $[0, \frac{1}{2}]$. As a benchmark, we consider the case in which there is no horizontal differentiation. Then, a symmetric Bertrand competition induces both schools to set price at the marginal cost, and the model returns to Spence's signaling game. In the least-cost separating equilibrium, all types except the lowest vertical type choose more education than the first-best, as they attempt to separate themselves from lower vertical types.

In Section 3, we consider the case in which a monopolist maximizes the joint profit of the two schools. We start with the observed case in which employers observe each school's tuition scheme. In the symmetric school-optimal separating equilibrium, when there is slight horizontal differentiation, the vertical market is partially covered and has two segments: in the fully covered range, all horizontal types purchase education; in the partially covered range, only those close to either school purchase education. Moreover, all vertical types except the highest one purchase less education than the first-best. This result stands in contrast to that of the Bertrand-Spence benchmark. The downward distortion results from the interaction of three forces: market penetration, screening and signaling. Since a higher type can benefit from his cost advantage over lower types, the monopolist has to leave information rent to the worker to incentivize truth-telling. In the fully covered range, since the market share is maximized, the marginal profit of education is unambiguously lower than the social surplus, thus the monopolist under-supplies education. In the partially covered range, in contrast, the monopolist can benefit from rent provision to gain market share. However, when there is slight horizontal differentiation, the screening effect is dominant, leading to a downward distortion. As the degree of horizontal differentiation rises, to maintain the market share in the partially covered range, the monopolist provides more rent to the worker by both raising the market coverage and offering more education to the worker.

When horizontal differentiation is sufficiently significant, the allocation depends critically on the intensity of signaling. As is pointed out by Lu (2018), in the monopoly observed case, signaling mitigates the screening distortion. This is because the worker's signaling incentive reduces his willingness to imitate lower types, and thus, the school leaves lower information rent to the worker than when signaling is absent. When signaling intensity is relatively low, screening outweighs signaling and market penetration, resulting in a downward distortion with a partially covered vertical market. When signaling intensity is at some intermediate level—when the worker's productivity and cost heterogeneity are equally significant—the monopolistic allocation achieves the first-best for all types. That is, the effects of signaling and market penetration exactly offset that of screening, thereby restoring the social optimum. In contrast, full-efficiency can never occur when signaling is absent, because otherwise the monopolist had to offer the worker a rent equal to the social surplus, leading to zero profit. Again, this is because the worker extracts higher information rent when signaling is absent. Then, for even higher signaling intensities, at the low end of the market where the monopolist charges very low price to increase market penetration, signaling outweighs screening, leading to over-education in this region.

Then, we turn to the unobserved case in which employers do not observe any school's tuition scheme. In the symmetric school-optimal separating equilibrium, the market coverage is lower, and education levels are more dispersed than in the observed case. Specifically, an interval of higher types choose more education than in the observed case, whereas the others choose less education than in the observed case. As in Lu (2018), this difference is driven by a *signal jamming effect*. Since employers cannot observe the actual cost of education, they will attribute a difference in education level to worker cost heterogeneity despite that tuition changes. Consequently, the worker's demand for education becomes more elastic than in the observed case. This provides the monopolist with an incentive to secretly supply more education. Suppose that, as in Lu (2018), there is no horizontal differentiation and thus the market contains only the fully covered range, then the vertical market is partially covered due to screening, and education levels are uniformly higher in the unobserved case than in the observed case. As the degree of horizontal differentiation rises, the partially covered range emerges, and the monopolist offers lower types more education to gain market share. However, due to incentive compatibility, doing so will provide higher types with higher information rent. Since in the unobserved case those higher types already obtain higher rent than in the observed case, the monopolist finds it unprofitable to offer those lower types the same education levels as in the observed case. Therefore, at any positive degree of horizontal differentiation, opposite to higher types, an interval of lower types obtain less education in the unobserved case than in the observed case, meaning that the market coverage is lower in the unobserved case. The length of such an interval is increasing in the degree of horizontal differentiation and vanishes as the degree approaches zero.

In Section 4, we consider duopoly in which each school maximizes own profit given the other's tuition scheme. Again, we start with the observed case. In contrast to monopoly, under duopoly, market competition results in a higher market coverage, higher education levels, and a higher equilibrium payoff to the worker. Intuitively, under duopoly, the two schools compete with each other in the fully covered range by providing the worker with more rent than in the monopoly case. This relaxes the incentive compatibility constraint for lower types. Specifically, each school fears less about allocating more education to lower types thereby providing higher types with more rent, as higher types will enjoy more rent anyway due to market competition. Therefore, the schools increase education supply for all participating types, and include some of those who are not served in the monopoly case.

In the unobserved case, from numerical computation, we obtain qualitatively identical results as in the observed case. However, the intuition is a bit subtler. Suppose that both schools retain the contract of the monopoly case, and thus, the labor market offers the same wage schedule. Then, given the other's tuition scheme, each school has an incentive to supply more education for two reasons. The first reason is the competition in rent provision between the two schools, as is suggested above. The second reason is that due to the signal jamming effect, each school has an incentive to secretly supply more education to "fool" the market thereby making a profitable deviation. Similarly, while higher types receive more education, so do lower types, as the incentive compatibility constraint relaxes. Thus, education levels are uniformly higher under duopoly than under monopoly; accordingly, the market coverage is higher under duopoly as well.

In Section 5, we conclude our paper. All omitted proofs are presented in the Appendix.

1.1 Related Literature

This paper is most closely related to three recent papers on nonlinear pricing: Rochet and Stole (2002), Yang and Ye (2008) and Lu (2018). Rochet and Stole (2002) studies both monopoly and duopoly nonlinear pricing in a Hotelling model. In this paper, horizontal types are interpreted as consumers' outside options, thereby giving rise to random participation. Their analysis focuses on the case in which the vertical market is always fully covered. As such, they show that under monopoly, there is either bunching or efficient allocation at the bottom. Under duopoly, when the market is fully covered, the equilibrium is such that both sellers offer an efficient cost-plus-fee price schedule.[1]

Yang and Ye (2008) complements Rochet and Stole (2002)'s analysis by focusing on the case in which the lowest participating type is endogenously determined. By doing so, they investigate the effects of horizontal differentiation and competition on the market coverage

[1]Armstrong and Vickers (2001) has obtained a similar result.

and quality distortion. The paper shows that under monopoly, the vertical market is always partially covered and bunching never happens. Moreover, quantity is downward distorted with efficiency achieved only on the top. When the market structure changes from monopoly to duopoly, the market coverage rises, and quality distortion decreases.

In contrast to Rochet and Stole (2002) and Yang and Ye (2008), the products in our model possess signaling value. Signaling affects the equilibrium allocation by mitigating the screening distortion. In particular, when the degree of horizontal differentiation is sufficiently high, for some certain level of signaling intensity, the monopolistic allocation can fully achieve the first-best; for higher signaling intensities, there is upward distortion at the low end of the vertical market. These results cannot be obtained in the other two papers in which signaling is absent. Recently, Ye and Zhang (2017) studies monopolistic nonlinear pricing with consumer entry. Different from the mechanism in our paper, they show that consumer entry can mitigate the screening distortion too. Under certain conditions, the first-best can also be achieved by the monopolistic allocation.

Lu (2018) studies monopolistic nonlinear pricing for products that provide signaling value to customers and assess how the transparency of pricing affects the degree of signaling and welfare. As in classic screening models, Lu (2018) makes two simplifying assumptions: the consumers possess one-dimensional private information, and their participation decisions are type-independent. In contrast, the current paper studies nonlinear pricing for horizontally differentiated products with signaling value. Therefore, the consumers have two-dimensional types and make type-dependent participation decisions. The results of Lu (2018) can be seen as the limit results of the current paper with respect to the degree of horizontal differentiation. Thus, there is no discontinuity in the results of Lu (2018) when we disturb the participation constraint somewhat. In addition, Rayo (2013) also consider monopolistic nonlinear pricing for signals, assuming that the seller's price schedule is observed by the receivers. Whereas we assume additive separability in the receivers' action and the consumers' type, Rayo's adopts a multiplicative structure, and thus, the seller's revenue depends on whether the allocation of signal is separating or pooling; this necessitates the use of novel screening techniques.

There are several other papers that study nonlinear pricing for both horizontally and vertically differentiate products in competitive settings. For example, Gilbert and Matutes (1993), Stole (1995), Verboven (1999), Villas-Boas and Schmidt-Mohr (1999), Ellison (2005), and Armstrong and Vickers (2001). Like Rochet and Stole (2002), all these papers assume that the vertical market is always covered, and thus, they preclude the effects of horizontal competition on the market coverage.

2 The Model

Players and actions. There are n schools, a worker and a competitive labor market. At the beginning of the game, each school i chooses a tuition scheme $T_i(z) : \mathbb{R}_+ \to \mathbb{R}_+$, where z stands for education level and $T_i(z)$ is the tuition at z. Then, observing all the tuition schemes, the worker chooses at most one school to attend, and upon attendance how much education to purchase from the school. The worker's education choice is thus characterized by which school he attends and how much education he chooses. Finally, the labor market offers the worker a wage equal to his expected productivity (see below).

The worker's productivity depends on his ability θ and education level z, irrespective of which school he attends.[2] The worker's ability θ is a random variable, which distributes uniformly over the unit interval: $\theta \sim U[0, 1]$. Let $Q(z, \theta)$ be the productivity of a worker with ability θ and education level z. Specifically, we assume that

$$Q(z, \theta) = \gamma\theta z + z,$$

where $\gamma > 0$ is a parameter. Thus, the productivity function is increasing in both arguments and is supermodular, meaning that both education and the worker's ability are productive, and complement each other. In addition, a worker with no education has zero productivity irrespective of his ability. This corresponds to the fact that many jobs require a minimal education level. For instance, a lawyer candidate must graduate from a law school, and medical school education is prerequisite for being a licensed practitioner of medicine.

The worker incurs a transportation cost if he attends a school. Specifically, the worker is located randomly and uniformly along a unit-length circle. The locations of all the schools split the circle evenly. Let d_i be the distance between the worker and school i. If the worker chooses to attend school i, then he incurs a transportation cost kd_i, where $k > 0$ is the unit transportation cost. Note that the worker's preference depends on his ability θ and his location that is summarized by $\{d_i\}$. Thus, the worker is characterized by a two-dimensional type $(\theta, \{d_i\})$, where the first preference parameter θ is called the worker's *vertical* type and the second parameter $\{d_i\}$ the worker's *horizontal* type, respectively, with both parameters independent of each other. Figure 1 illustrates the locations of two schools and the worker in a duopoly education market.

Information. The worker's education choice is publicly observed. Whereas the distribution of the worker's type is common knowledge, neither the schools nor the labor market observes

[2] That is, the education provided by each school is equally productive. Nonetheless, the wage offered by the labor market may still depend on which school the worker attends.

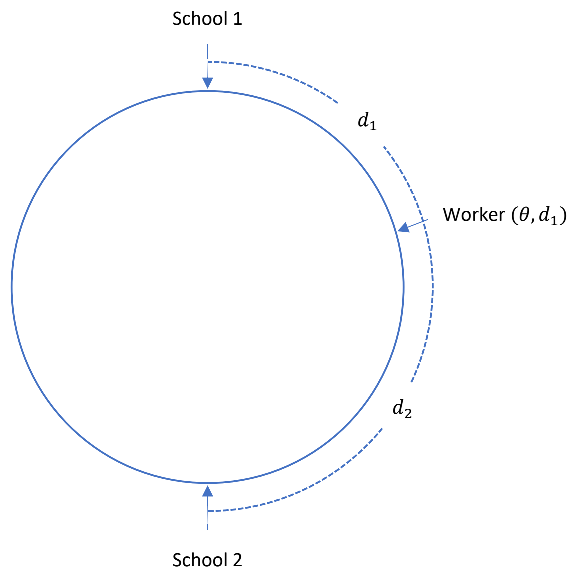

Figure 1: **A Duopoly Education Market.**

the worker's type. In this paper, for each market structure we consider, we study two variants of the model: in the *observed* case, all the tuition schemes are observed by the labor market; in the *unobserved* case, no tuition scheme is observed by the labor market. In each case, based on the available information, the labor market announces and commits to a wage schedule $W_i(z) : \mathbb{R}_+ \to \mathbb{R}_+$ for each school i's student.

Payoffs. We normalize each school's production cost to zero. Thus, school i's per-customer profit equals the tuition revenue T_i. If a type-$(\theta, \{d_i\})$ worker attends school i and chooses education level z, then he obtains a *gross* utility given by

$$V_i(z, \theta) = W_i(z) - T_i(z) - C(z, \theta),$$

and accordingly a *net* utility given by

$$U_i(z, \theta, d_i) = V_i(z, \theta) - kd_i,$$

where $C(z, \theta)$ is the worker's cost of effort for education. Specifically, we assume that

$$C(z, \theta) = z^2 + (1 - \theta)z.$$

Note that $C(z, \theta)$ is increasing and strictly convex in z, and that $C(0, \theta) = 0$ for any θ. More importantly, the standard *single-crossing property* holds: $C_{z\theta}(z, \theta) < 0$ if $z > 0$. This condition captures the feature that a higher-ability worker has lower marginal effort costs than a lower-ability worker. We also assume that the worker can obtain a zero-utility outside option by neither attending school nor entering the labor market.

First-best benchmark. Define $S(z, \theta)$ as the social surplus function (net from the transportation cost). Then, we have

$$S(z, \theta) = Q(z, \theta) - C(z, \theta) = (\gamma + 1)\theta z - z^2$$

It follows that the first-best education level is given by $z^{fb}(\theta) = \frac{(\gamma+1)\theta}{2}$. Substituting $z^{fb}(\theta)$ into $S(z, \theta)$, we have $S^{fb}(\theta) = \frac{(\gamma+1)^2\theta^2}{4}$.

Equilibrium. Throughout the paper, we use *symmetric perfect Bayesian equilibrium* as the solution concept. Specifically, in the observed case, an equilibrium consists of each school i's tuition scheme T_i^o and conditional on any tuition scheme profile $\{T_i\}$, the worker's education choice $z_i^o(\theta; \{T_i\})$ and the labor market's wage schedule $W_i^o(z; \{T_i\})$ for each i, such that

(*i*) For each $\{T_i\}$: (*a*) given $W_i^o(z; \{T_i\})$, $z_i^o(\theta; \{T_i\})$ maximizes $U_i(z, \theta, d_i)$; (*b*) $W_i^o(z; \{T_i\}) = \mathbb{E}[Q(z, \theta)|z_i^o(\theta; \{T_i\})]$ such that the labor market's posterior belief about the worker's ability, or simply *the market belief*, is updated using Bayes' rule.

(*ii*) Given $z_i^o(\theta; \{T_i\})$ and $\{T_{-i}^o\}$, T_i^o maximizes the school's expected profit, i.e.,

$$T_i^o \in \arg\max_{T_i} \int_0^1 T_i(z_i^o(\theta; \{T_i\}))d\theta$$

subject to that $T_j^o = T_i^o$ for any $j \neq i$.

In the unobserved case, the market belief is independent of the actual tuition schemes but is conditional on *conjectured* schemes. We assume that the conjecture is symmetric across schools, and in equilibrium, it is correct. In this case, an equilibrium consists of each school i's tuition scheme T_i^u and the associated wage schedule W^u (more precisely, $W^u(z; \{T_i^u\})$), and conditional on any profile $\{T_i\}$, an education function $z_i^u(\theta; \{T_i\})$ for each i, such that

(*i*) Given W^u, for each $\{T_i\}$, $z_i^u(\theta; \{T_i\})$ maximizes $U_i(z, \theta, d_i)$; $W^u = \mathbb{E}[Q(z, \theta)|z_i^u(\theta; \{T^u\})]$ such that the market belief is updated using Bayes' rule.

(*ii*) Given $z_i^u(\theta; \{T_i\})$ and $\{T_{-i}^u\}$, T_i^u maximizes the school's expected profit, i.e.,

$$T_i^u \in \arg\max_{T_i} \int_0^1 T_i(z_i^u(\theta; \{T_i\}))d\theta$$

subject to that $T_j^u = T_i^u$ for any $j \neq i$.

Note that the equilibrium conditions have one important difference between the observed and unobserved case: in the unobserved case, the market belief needs to be correct only on the equilibrium path, whereas in the observed case, the market belief has to be correct following every tuition scheme profile that is chosen by the schools.

Note too that all schools are symmetric and the worker's location distributes uniformly along the circle. Since we consider symmetric equilibrium, following Yang and Ye (2008), we claim without argument that the analysis for a n-school oligopoly model can be translated into that of a duopoly model if we normalize k to $k' = 2k/n$.[3] Since we consider any $k > 0$, it is without loss of generality to focus on the duopoly model. Thus, in the subsequent, we consider a duopoly education market as is depicted in Figure 1. As a result, the worker's horizontal type can be simply characterized by d_i, $i = 1, 2$.

Equilibrium selection. Due to the flexibility of off-path belief, there possibly exist multiple equilibria even though we consider symmetric equilibrium. Following Lu (2018), for both the observed and unobserved case, we focus on the *school-optimal separating equilibrium*; that is, the equilibrium that yields the highest payoff for the schools, provided that on the equilibrium path, $z(\theta)$ is one-to-one if $z(\theta) > 0$.

2.1 Direct Mechanisms

It is well known that in common agency games, it is no longer without loss of generality to restrict attention to direct mechanisms by applying the revelation principle.[4] In this regard, following Rochet and Stole (2002), we restrict our attention to deterministic contracts.[5] Note that the worker's gross utility, upon purchasing from school i, depends only on his vertical type θ. Thus, it is without loss of generality to consider direct mechanisms such that the allocation depends only on the vertical type the worker reports to a school. For brevity, in the subsequent, we often interchange *vertical type* and *type*, provided there is no confusion.

Hence, for both the observed and unobserved case, it is without loss of generality to adjust the timing as follows. First, each school i offers a contract $\langle z_i(\theta), T_i(z)\rangle$ to the worker. Then, the labor market posts a wage schedule $W_i(z)$ for each school i's student based on the

[3] See Section 5 of Yang and Ye (2008) for greater details.

[4] Martimort and Stole (2002) demonstrates an example in which the revelation principle may fail when competing principals deviate to more complicated mechanisms that incorporate off-path messages. The reason for such failure, as indicated by McAfee (1993), is that the mechanisms offered by other competing principals may also be the agent's private information when making his decision. This implies that such private information can potentially be used by competing principals in designing revelation mechanisms.

[5] see Rochet and Stole (2002) for a detailed discussion of restrictions due to focusing on deterministic contracts and excluding the possibility of random contracts. In contrast, Manelli and Vincent (2006) and Thanassoulis (2004) consider random contracts for indivisible goods in multi-dimensional screening games.

information available: in the observed case, it observes all the contracts; in the unobserved case, it does not observe any contract. Finally, the worker chooses at most one school to attend, and upon attendance he reports his type to only this school. If the worker chooses to attend school i and reports a type $\hat{\theta}$, then he obtains education level $z_i(\hat{\theta})$, pays tuition $T_i(z_i(\hat{\theta}))$ and then receives a wage $W_i(z_i(\hat{\theta}))$.

Worker's problem. In both cases, given each school i's contract $\langle z_i(\theta), T_i(z)\rangle$ and the associated wage schedule $W_i(z)$, a type-θ worker chooses some school i to attend, and upon attendance a report $\hat{\theta}_i$ to the school to maximize his net utility:

$$U_i(\hat{\theta}_i, \theta, d_i) = W_i(z_i(\hat{\theta}_i)) - T(z_i(\hat{\theta}_i)) - C(z_i(\hat{\theta}_i), \theta) - kd_i.$$

The mechanism $\{\langle z_i(\theta), T_i(z)\rangle, W_i(z)\}$ is *incentive compatible* (IC) if the worker is willing to truthfully report his type and is *individually rational* (IR) if the worker obtains a non-negative utility level by attending school i. A type-θ worker's equilibrium payoff is given by $U(\theta, d_i) \equiv \max_i U_i(\theta, \theta, d_i)$, and the corresponding gross utility by $V(\theta) \equiv V_i(\theta, \theta)$.

School's problem. In the observed case, given the other school's contract, each school chooses a contract to maximize its expected profit subject to the IC and IR constraints, and the correctness of the market belief. In the unobserved case, since the market's inference is independent of the schools' choices, given the wage schedules and the other school's contract, each school chooses a contract to maximize its expected profit subject to only the IC and IR constraints.

2.2 Preliminary Analysis

For both the observed and unobserved case, an allocation $\langle z(\theta), U(\theta, d_i)\rangle$ is *implementable* if it is incentive compatible and individually rational. Since the worker's net utility is separable in z and d_i, an allocation $\langle z(\theta), U(\theta, d_i)\rangle$ is incentive compatible if and only if the allocation of education level and gross utility, $\langle z(\theta), V(\theta)\rangle$, is incentive compatible. Appealing to Mas-Colell, Whinston, and Green (1995, Proposition 23.D.2), we can characterize the set of all incentive compatible allocations by the following lemma.

Lemma 1. *In both cases, an allocation $\langle z(\theta), V(\theta)\rangle$ is incentive compatible if and only if*

(i) $z(\theta)$ is non-decreasing.

(ii) Define $\theta_0 \equiv \inf\{\theta | z(\theta) > 0\}$; then, for $\theta > \theta_0$,

$$V(\theta) = V(\theta_0) + \int_{\theta_0}^{\theta} -C_\theta(z(s), s)ds = V(\theta_0) + \int_{\theta_0}^{\theta} z(s)ds.$$

Note that if the lowest participating type θ_0 is an interior type, i.e., $\theta_0 \in (0,1)$, then by continuity, $V(\theta_0)$ is optimally set to 0.[6] Following Armstrong and Vickers (2001), we think each school as directly providing utility to the worker. Let $V_i(\theta)$ be school i's *rent provision* to a type-θ worker. According to Lemma 1, if school i's allocation $\langle z_i(\theta), V_i(\theta)\rangle$ is incentive compatible, then we must have

$$V_i'(\theta) = z_i(\theta),\ T_i(z_i(\theta)) = W_i(z_i(\theta)) - C(z_i(\theta),\theta) - V_i(\theta).$$

This means that any incentive compatible contract can be characterized by a rent provision schedule $V_i(\theta)$, and thus, individual rationality holds if and only if $V_i(\theta) - kd_i \geq 0$.

Given the rent provision schedules $\{V_i(\theta)\}$, $i = 1, 2$, the worker decides whether to attend school, if so, which school to attend. If a type-(θ, d_i) worker chooses to attend school i, then we must have

$$V_i(\theta) - kd_i \geq \max\left\{0, V_{-i}(\theta) - k(\frac{1}{2} - d_i)\right\}.$$

This is equivalent to

$$d_i \leq \min\left\{\frac{V_i(\theta)}{k}, \frac{1}{4} + \frac{V_i(\theta) - V_{-i}(\theta)}{2k}\right\} := s_i(\theta).$$

Hence, school i's market share for each vertical type θ is given by $2s_i(\theta)$. Since $V_i(\theta)$ is increasing in θ, there is a cutoff type θ_1 above which the horizontal market is fully covered; that is, if the worker has a vertical type $\theta \in [\theta_1, 1]$, then he attends school irrespective of his horizontal type d_i. As such, we call the interval $[\theta_1, 1]$ *the competition range*, as in Yang and Ye (2008). In contrast, for $\theta \in [\theta_{0_i}, \theta_1)$,[7] the horizontal market is partially covered; thus, we call the interval $[\theta_{0_i}, \theta_1)$ *the local monopoly range*. Note that θ_1 is endogenously given by

$$V_1(\theta_1) + V_2(\theta_1) = \frac{k}{2}.$$

Then, we can represent the schools' expected payoffs with respect to the rent provision schedules. Given V_{-i}, school i's expected profit equals twice

$$\int_{\theta_{0_i}}^{1} \left[W_i(z_i(\theta)) - C(z_i(\theta),\theta) - V_i(\theta)\right] s_i(\theta) d\theta.$$

[6] $V(\theta_0)$ is not necessarily 0 if θ_0 is the lowest type $\underline{\theta}$. In general, if the lowest type can generate positive social surplus, then the school may leave a positive "rent" $V(\underline{\theta})$ to type $\underline{\theta}$, in order to gain the market share.

[7] Here, θ_{0_i} is also a control variable chosen by school i such that any type $\theta \in [0, \theta_{0_i})$ is excluded from attending school i.

By decomposing the above integral into the local monopoly range and the competition range, we have that school i's expected profit is twice

$$\int_{\theta_{0_i}}^{\theta_1} \left[W_i(z_i(\theta)) - C(z_i(\theta),\theta) - V_i(\theta)\right] \frac{V_i(\theta)}{k} d\theta + \int_{\theta_1}^{1} \left[W_i(z_i(\theta)) - C(z_i(\theta),\theta) - V_i(\theta)\right] \cdot \left[\frac{1}{4} + \frac{V_i(\theta) - V_{-i}(\theta)}{2k}\right] d\theta. \tag{2.1}$$

In the observed case, correctness of the market belief means that $W_i(z) = \mathbb{E}[Q(z,\theta)|z_i(\theta)]$ for any implementable allocation $z_i(\theta)$ that the school chooses. Then, from the law of total expectation, (2.1) can be rewritten as

$$\int_{\theta_{0_i}}^{\theta_1} \left[S(z_i(\theta),\theta) - V_i(\theta)\right] \frac{V_i(\theta)}{k} d\theta + \int_{\theta_1}^{1} \left[S(z_i(\theta),\theta) - V_i(\theta)\right] \cdot \left[\frac{1}{4} + \frac{V_i(\theta) - V_{-i}(\theta)}{2k}\right] d\theta. \tag{2.2}$$

Thus, given V_{-i}, school i's problem is to choose a contract $\langle z_i(\theta), V_i(\theta)\rangle$ to maximize (2.2), subject to $z_i(\theta)$ being non-decreasing and that $V_i'(\theta) = z_i(\theta)$. If the solution to this program is identical for schools $i = 1, 2$ with $z_i(\theta)$ being increasing over $[\theta_0, 1]$, then we obtain a symmetric school-optimal separating equilibrium.

In the unobserved case, given V_{-i} and W_i, each school i's problem is to choose a contract $\langle z_i(\theta), V_i(\theta)\rangle$ to maximize (2.1), subject to $z_i(\theta)$ being non-decreasing and that $V_i'(\theta) = z_i(\theta)$. Without loss of generality, assume that each school chooses a contract, while simultaneously, the labor market chooses a corresponding wage schedule. Then, the equilibrium conditions can be simplified as follows: for each school $i = 1, 2$, (i) given V_{-i}^u and W_i^u, $\langle z_i^u(\theta), V_i^u(\theta)\rangle$ solves school i's problem; (ii) $W_i^u(z) = \mathbb{E}[Q(z,\theta)|z_i^u(\theta)]$ such that the market belief is updated using Bayes' rule. In the case of multiple equilibria, we select a symmetric school-optimal separating equilibrium.

2.3 A Bertrand-Spence Benchmark: $k = 0$

As a benchmark, we consider a duopoly education market in which the worker can attend any school at zero transportation cost, i.e., $k = 0$. In this case, a symmetric Bertrand competition induces both schools to set tuition at the marginal cost which is zero. The model is thus translated to a Spence's signaling game (Spence 1973). An equilibrium of this game consists of an education function $z^s(\theta)$ and a wage schedule $W^s(z)$, such that (i) given $W^s(z)$, $z^s(\theta)$ maximizes $U(z,\theta)$; (ii) $W^s(z) = \mathbb{E}[Q(z,\theta)|z^s(\theta)]$ with the market belief updated using Bayes' rule. As in Lu (2018), we focus on the *least-cost separating equilibrium* such that $z^s(0) = z^{fb}(0)$. Applying Lu (2018, Proposition 3.1), we have that the least-cost separating equilibrium exists, such that

$$z^s(\theta) = \frac{(2\gamma+1)\theta}{2} \text{ on } [0,1],$$
$$W^s(z) = \frac{2\gamma}{2\gamma+1}z^2 + z \text{ on } [0, \gamma+\frac{1}{2}].$$

It follows that $z^s(\theta) > z^{fb}(\theta)$ for all $\theta > 0$. The intuition is well understood. Specifically, since the worker's ability is private information, he attempts to separate himself from lower ability workers by acquiring more education, thereby leading to over-education. Given the analytical solution of $z^s(\theta)$, the signaling effect is explicitly given by

$$Q_\theta(z^s(\theta), \theta) \cdot \theta^s(z) = \frac{2\gamma z}{2\gamma+1} > 0.$$

The signaling effect reflects the feature that a higher education level makes the labor market regard the worker as having higher ability.

Furthermore, we can parameterize the intensity of signaling in this model. Let us define the intensity of signaling to be the ratio of the over-invested education in Spence's model, i.e., $z^s(\theta) - z^{fb}(\theta)$, to the first best education level $z^{fb}(\theta)$ for $\theta > 0$. Substituting, we have

$$\frac{z^s(\theta) - z^{fb}(\theta)}{z^{fb}(\theta)} = \frac{\gamma}{\gamma+1}.$$

Clearly, the intensity of signaling is increasing in the parameter γ. To see the idea, note that the larger γ is, the stronger complementarity between the worker's ability and education is. Due to the signaling effect, a higher education level induces the labor market to regard the worker as having higher ability; hence, if ability complements education to a larger extent, the marginal benefit of education will be even higher, thereby enhancing signaling through education. Consequently, over-education will be more serious.

3 Monopoly

In this section, as a well-controlled benchmark, we consider a monopoly education market in which both schools are owned by a monopolist. The monopolist's objective is thus to maximize the joint profit of the two schools. Since the distribution of the worker's type is uniform and the schools' locations and technologies are symmetric, we assume that for both the observed and unobserved case, the two schools offer an identical contract to the worker, thereby resulting in symmetric market shares. In what follows, we start our analysis with the observed case.

3.1 The Observed Case

Since both schools offer the same contract, we can drop the subscripts to simplify (2.2). The monopolist's problem can be stated as

$$\max \underbrace{\int_{\theta_0}^{\theta_1} [S(z(\theta),\theta) - V(\theta)] \frac{V(\theta)}{k} d\theta}_{\text{Phase I: partially covered range}} + \underbrace{\int_{\theta_1}^{1} [S(z(\theta),\theta) - V(\theta)] \frac{1}{4} d\theta}_{\text{Phase II: fully covered range}}$$

$$s.t.\ V'(\theta) = z(\theta),\ z'(\theta) \geq 0,\ V(\theta_1) = \frac{k}{4}.$$

If further $\theta_0 \in (0, 1]$, then we have $V(\theta_0) = 0$; otherwise, we have to choose $V(\theta_0)$ optimally.

As is standard in the literature, we solve the above program by relaxing the monotonicity constraint of $z(\theta)$ first and verity it ex post to justify the approach. The monopolist's problem is a two-phase optimal control problem: in Phase I, the horizontal market is partially covered; in Phase II, in contrast, the horizontal market is fully covered. Define the *Hamiltonian* of the two phases as follows:

$$H_1(z, V, \lambda, \theta) = [S(z, \theta) - V]\frac{V}{k} + \lambda z = [(\gamma + 1)\theta z - z^2 - V]\frac{V}{k} + \lambda z,$$
$$H_2(z, V, \lambda, \theta) = [S(z, \theta) - V]\frac{1}{4} + \lambda z = [(\gamma + 1)\theta z - z^2 - V]\frac{1}{4} + \lambda z,$$

where z is a control variable, V is a state variable and λ is the associated adjoint variable. From the Maximum Principle,[8] if $\langle z^*(\theta), V^*(\theta)\rangle$ solves the monopolist's problem, then for each phase $i = 1, 2$, we must have

$$z^*(\theta) = \arg\max_z H_i\left(z, V^*(\theta), \lambda(\theta), \theta\right),$$
$$\dot{\lambda}(\theta) = -\frac{\partial}{\partial V} H_i\left(z^*(\theta), V^*(\theta), \lambda(\theta), \theta\right),$$

combined with the transversality condition $\lambda(1) = 0$.

It follows that Phase I can be characterized by the following second order autonomous ordinary differential equation (ODE):

$$(\gamma + 3)V - 2\ddot{V}V - \dot{V}^2 = 0. \tag{3.1}$$

To solve (3.1), we first consider the case in which the vertical market is partially covered; that is, the optimal lowest participating type $\theta_0^* \in (0, 1]$, and thus, $V(\theta_0^*) = 0$. Given this

[8]See Seierstad and Sydsaeter (1986) for details.

boundary condition, it can be proved that the unique solution to (3.1) is given by[9]

$$V^*(\theta) = \frac{\gamma+3}{8}(\theta-\theta_0^*)^2,\ z^*(\theta) = \frac{\gamma+3}{4}(\theta-\theta_0^*).$$

Analogously, in Phase II, we obtain the ODE: $\ddot{V} = \dot{z} = \frac{\gamma+2}{2}$. Moreover, the transversality condition $\lambda(1) = 0$ implies that $z(1) = \frac{\gamma+1}{2}$. Thus, the solution to Phase II is given by

$$V^*(\theta) = \frac{(\gamma+2)\theta^2}{4} - \frac{\theta}{2} + \beta(\theta_1^*),\ z^*(\theta) = \frac{(\gamma+2)\theta}{2} - \frac{1}{2},$$

where $\beta(\theta_1^*)$ depends on the optimal switching type θ_1^* which remains to be determined. Note that in both phases, $z(\theta)$ is increasing in θ. Thus, the monotonicity constraint is satisfied automatically, meaning that a symmetric school-optimal separating equilibrium exists.

To determine θ_1^* and thus θ_0^*, we impose the smooth pasting conditions: $V(\theta_1^{*-}) = V(\theta_1^{*+})$ and $z(\theta_1^{*-}) = z(\theta_1^{*+})$.[10] Combined with the condition $V(\theta_1^*) = \frac{k}{4}$, θ_1^* and θ_0^* are thus given by

$$\theta_0^* = \frac{1}{\gamma+2} - \frac{(\gamma+1)\sqrt{2(\gamma+3)k}}{2(\gamma+2)(\gamma+3)},\ \theta_1^* = \frac{1}{\gamma+2} + \frac{\sqrt{2(\gamma+3)k}}{2(\gamma+2)}. \tag{3.2}$$

It thus follows that for $\theta \in [\theta_1^*, 1]$, $V(\theta)$ is given by

$$V^*(\theta) = \frac{k}{4} + (\theta-\theta_1^*)\left[\frac{(\gamma+2)(\theta+\theta_1^*)}{4} - \frac{1}{2}\right].$$

Note that θ_0^* is an interior solution if and only if $k < \frac{2(\gamma+3)}{(\gamma+1)^2}$. We shall consider two cases. First, $\gamma \leq 1$. In this case, when $k \geq \frac{2(\gamma+3)}{(\gamma+1)^2}$, we have $\theta_1^* \geq 1$, meaning that Phase II is never entered. Then, θ_0^* is pinned down by the transversality condition $\lambda(1) = 0$, such that $\theta_0^* = \frac{1-\gamma}{\gamma+3} \geq 0$, with equality holding at $\gamma = 1$ only. Hence, if $\gamma < 1$, then θ_0^* is always an interior solution; in addition, Phase I (the partially covered range) always exists, whereas Phase II (the fully covered range) exists only if $k < \frac{2(\gamma+1)^2}{\gamma+3}$. Second, $\gamma > 1$. In this case, when $k = \frac{2(\gamma+3)}{(\gamma+1)^2}$, we have $\theta_1^* < 1$, meaning that Phase II always exists. Thus, if $k \geq \frac{2(\gamma+3)}{(\gamma+1)^2}$, then $\theta_0^* \leq 0$; that is, the vertical market is fully covered. As a result, $V(\theta)$ is free at the lowest type $\theta = 0$ and the boundary condition $V(0) = 0$ does not necessarily hold. Since such a case is more complicated, we postpone further analysis until we have summarized the results of the case in which the vertical market is partially covered.

Suppose that $\gamma \leq 1$, then the monopolist's optimal symmetric contract exists and has been characterized in the above analysis. Let $\langle z^{m_o}(\theta), V^{m_o}(\theta)\rangle$ be the equilibrium contract

[9] Rochet and Stole (2002) shows that if a convex solution to (3.1) exists, then given specific boundary conditions, it is unique. See Rochet and Stole (2002, Appendix, p. 304-305) for details.

[10] The smooth pasting conditions are implied by the Weierstrass-Erdmann necessary condition.

in the observed case under monopoly, and $\theta_0^{m_o}$ and $\theta_1^{m_o}$ be the lowest participating type and switching type, respectively. Then, we have that $z^{m_o}(\theta)$ is increasing on $[\theta_0^{m_o}, 1]$; in particular, if $\gamma < 1$, then $\theta_0^{m_o} > 0$ always holds. We have thus obtained the school-optimal separating equilibrium. Indeed, this equilibrium is the one that yields the highest equilibrium payoff for the monopolist among all equilibria. To summarize, we have the following proposition:

Proposition 1. *Suppose that $\gamma \leq 1$, then in the observed case under monopoly, the symmetric school-optimal separating equilibrium exists. Specifically, for $k \in \left(0, \frac{2(\gamma+1)^2}{(\gamma+3)}\right)$,*

$$z^{m_o}(\theta) = \begin{cases} \frac{\gamma+3}{4}(\theta - \theta_0^{m_o}) & \text{if } \theta_0^{m_o} \leq \theta < \theta_1^{m_o} \\ \frac{(\gamma+2)\theta}{2} - \frac{1}{2} & \text{if } \theta_1^{m_o} \leq \theta \leq 1, \end{cases}$$

where $\theta_0^{m_o}$ and $\theta_1^{m_o}$ are given by θ_0^* and θ_1^* in (3.2), respectively. For $k \geq \frac{2(\gamma+1)^2}{(\gamma+3)}$,

$$z^{m_o}(\theta) = \frac{\gamma+3}{4}\left(\theta - \theta_0^{m_o}\right), \text{ if } \theta_0^{m_o} \leq \theta \leq 1,$$

where $\theta_0^{m_o} = \frac{1-\gamma}{\gamma+3}$. If $\gamma < 1$, then for any $k > 0$, $\theta_0^{m_o} > 0$ and $z^{m_o}(\theta) \leq z^{fb}(\theta)$ on $[0,1]$ with equality holding at $\theta = 1$ only; if $\gamma = 1$, then for $k \geq \frac{2(\gamma+1)^2}{(\gamma+3)}$, $z^{m_o}(\theta) = z^{fb}(\theta)$ on $[0,1]$.

The monopolist's optimal contract has two noticeable features. First, when $\gamma < 1$, there is always under-education on both the extensive and intensive margin. Specifically, there is always a positive measure of vertical types who are excluded from education, i.e., $\theta_0^{m_o} > 0$. In addition, for all but the highest vertical type, education level is downward distorted, i.e., $z^{m_o}(\theta) < z^{fb}(\theta)$ on $[\theta_0^{m_o}, 1)$. This result stands in contrast to that of Spence's model in which there is always over-education. Second, perhaps it is more striking that when $\gamma = 1$, if the market contains only the partially covered range, then the monopoly optimal contract in fact achieves the first-best! In contrast, Lu (2018) studies monopolistic nonlinear pricing for signals with deterministic participation and find that the optimal contract can achieve the first-best asymptotically.[11] In addition, Rochet and Stole (2002) and Yang and Ye (2008) study monopolistic nonlinear pricing for non-signaling goods with random participation, and both papers find that the optimal contract always exhibits a downward distortion with efficiency achieved only on the boundary.[12]

The above features result from the interaction between three forces: market penetration, the monopolist's screening and the worker's signaling. To be specific, let us first consider the

[11] Using a numerical example similar to the current model, Lu (2018) shows that $z^{m_o}(\theta)/z^{fb}(\theta) \to 1$ as $\gamma \to \infty$. See Section 4.2 of Lu (2018) for details.

[12] See Rochet and Stole (2002, Proposition 4) and Yang and Ye (2008, Proposition 1).

fully covered range. Note that the monopolist's market share is already maximized for each vertical type, thus it cannot benefit from supplying more rent to gain market share. Since a higher vertical type can benefit from his cost advantage over lower types, the monopolist has to leave information rent to the worker to incentivize truth-telling. This implies that the marginal profit of education is unambiguously less than the marginal social surplus in the fully covered range, and thus, the monopolist under-supplies education. In Lu (2018), the market contains only the fully covered range, and hence, there is always a downward distortion with efficiency achieved only on the top.

In contrast, in the partially covered range, the monopolist can benefit from rent provision to obtain market share. In this case, increasing education supply has two opposite effects. On one hand, it reduces per-customer profit by providing the worker with more information rent. On the other hand, a larger rent also results in a larger market share. The optimal allocation rule thus must balance these two opposite effects. But the question is: why does under-education always occur in the partially covered range when $\gamma < 1$, whereas the social optimum can be fully achieved when $\gamma = 1$?

To answer this question, one should understand the effects of signaling on the optimal allocation. As is pointed out by Lu (2018), in the observed case under monopoly, signaling can mitigate the screening distortion. To see this, note that given the monopolist's tuition scheme, the subgame is indeed a Spence's signaling game as if the worker's cost function was given by $T(z) + C(z, \theta)$; thus, the signaling effect induces the worker to "over-invest" in education in terms of total cost. The signaling incentive reduces the worker's willingness to intimate lower types, therefore, the worker extracts less information rent than when signaling is absent. To illustrate, suppose that the labor market can observe the worker's ability, thereby eliminating signaling.[13] As a result, we return to Rochet and Stole (2002) or Yang and Ye (2008). In this case, the IC constraint is given by $V'(\theta) = S_\theta(z(\theta), \theta)$. In contrast, in the current environment, it is given by $V'(\theta) = C_\theta(z(\theta), \theta) < S_\theta(z(\theta), \theta)$. This reveals that the monopolist leaves less information rent to the worker when signaling is present, and thus, signaling can mitigate the screening distortion.

Recall that γ measures signaling intensity: the larger γ is, the more intense signaling is. Thus, if γ is relative big, then the screening distortion can be mitigated to a relatively great extent. Specifically, if the market has only the partially covered range, then it can be easily verified that the ratio $z^{m_o}(\theta)/z^{fb}(\theta)$ is increasing in γ for all θ, and arrives at 1 when $\gamma = 1$; that is, the effects of market penetration and signaling can exactly offset that of screening,

[13]Signaling is eliminated because the worker's wage always equals his actual productivity.

thereby restoring the first-best allocation.

At first glance, it may be surprising that under information asymmetry the monopolistic profit-maximizing pricing can be welfare-maximizing. To see the intuition, note that if the market contains only the partially covered range, then the marginal profit of each type θ is given by $[S(z(\theta),\theta) - V(\theta)]V(\theta)/k$. Suppose that the monopolist can observe the worker's vertical type, then the optimal contract $\langle z^*(\theta), V^*(\theta)\rangle$ is simply that $z^*(\theta) = z^{fb}(\theta)$ and $V^*(\theta) = S(z^{fb}(\theta),\theta)/2$. This is because $V = S/2$ maximizes the marginal profit for any S and $z^*(\theta) = z^{fb}(\theta)$ maximizes $S(z,\theta)$. However, even without the ability to contract on θ, such an allocation can still be implemented without violating the IC constraint $V'(\theta) = z(\theta)$ when $\gamma = 1$. In contrast, in the contracting problems of Rochet and Stole (2002) and Yang and Ye (2008), the first-best can never be fully achieved. Suppose not, then $z^*(\theta) = z^{fb}(\theta)$ on $[0,1]$. It follows from the envelope theorem and the associated IC constraint that

$$S^*(\theta) = S(z^{fb}(\theta),\theta) = \int_0^\theta S_\theta(z^{fb}(s),s)ds = V^*(\theta).$$

Thus, the marginal profit is always zero, which cannot be optimal. Again, this is due to that the monopolist supplies more information rent when signaling is absent.

It is worth noting that when $\gamma = 1$, the worker's productivity and cost heterogeneities are equally significant (i.e., $Q_{z\theta} = C_{z\theta}$). Since the lowest type is totally unproductive, at the social optimum, each type's social surplus is exactly twice his information rent, as is shown in the above. Thus, the first-best can be fully achieved if the market has only the partially covered range. Moreover, it can be easily verified that in equilibrium the signaling effect that is measured by $Q_\theta \cdot \theta'(z)$ equals the marginal tuition $T'(z)$.[14] This implies that when the market contains only the partially covered range, the monopolistic optimal tuition scheme levies *Pigovian tax* on signaling, which undoes the signaling effect and thus restores the first-best. In contrast, in Lu (2018), the market has only the fully covered range; thus, the optimal tuition scheme "over-taxes" signaling and leads to a downward distortion.

Going forward, we turn to the case in which $\gamma > 1$. When $k < \frac{2(\gamma+3)}{(\gamma+1)^2}$, we have $\theta_0^* > 0$, i.e., the vertical market is partially covered. Thus, the equilibrium contract is characterized by Proposition 1. In contrast, when $k \geq \frac{2(\gamma+3)}{(\gamma+1)^2}$, we have $\theta_0^* \leq 0$, i.e., the vertical market is fully covered. In this case, the equilibrium contract in the fully covered range is characterized by

[14]When $\gamma = 1$, if in equilibrium only the partially covered range exists, then $W(z)$ and $T(z)$ are given by

$$W(z) = z^2 + z,\ T(z) = \frac{1}{2}z^2.$$

Thus, we have $Q_\theta \cdot \theta'(z) = 2z - \theta(z) = z = T'(z)$, as $z^{m_o}(\theta) = z^{fb}(\theta) = \theta$.

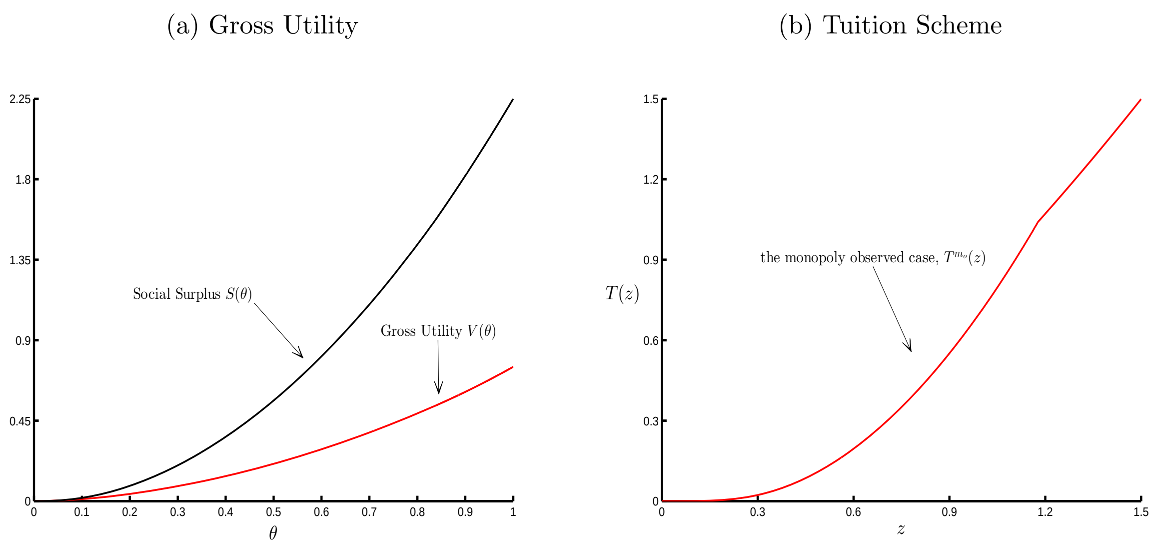

Figure 2: **A Numerical Solution.** This figure assumes that $\gamma = 2$ and $k = 2$.

Proposition 1 too; thus, the boundary conditions $V(\theta_1^*) = \frac{k}{4}$ and $\dot{V}(\theta_1^*) = \frac{(\gamma+2)\theta_1^*-1}{2}$ remain. However, the initial state $V(0)$ is now free. This means that the adjoint variable λ satisfies: $\lambda(0) = 0$.[15] It follows that efficiency occurs at the bottom, i.e., $z^*(0) = z^{fb}(0) = 0$.[16] This yields an extra boundary condition: $\dot{V}(0) = 0$. In summary, when $k \geq \frac{2(\gamma+3)}{(\gamma+1)^2}$, the optimal contract in the partially covered range is given by the solution to the following problem:

$$(\gamma + 3)V - 2\ddot{V}V - \dot{V}^2 = 0,$$

$$s.t.\ \dot{V}(0) = 0,\ V(\theta_1) = \frac{k}{4},\ \dot{V}(\theta_1) = \frac{(\gamma + 2)\theta_1 - 1}{2}.$$

Note that this is not a standard boundary value problem (BVP), as the boundary conditions involve an endogenous endpoint θ_1. As far as we know, no existing BVP theorem can be applied directly to show the existence and uniqueness of the solution to this problem, not mention deriving an analytical solution. In this regard, we solve the problem using numerical methods.[17] In Figure 2, panel (a) depicts a convex solution $V(\theta)$ for a sufficiently large k, and panel (b) depicts the associated equilibrium tuition scheme $T^{m_0}(z)$.

The equilibrium outcome has a salient feature: that is, when the unit transportation cost k is sufficiently large, over-education occurs at the low end of the spectrum of θ. Moreover, if the market contains the fully covered range, then there exists a cutoff type such that all lower vertical types obtain more education than the first-best, whereas the others obtain

[15]See Seierstad and Sydsaeter (1986, p. 185-186) for details

[16]Rochet and Stole (2002) provides an intuitive discussion about efficiency at the bottom in its Appendix.

[17]The MATLAB code for all numerical calculations of this paper is available upon request.

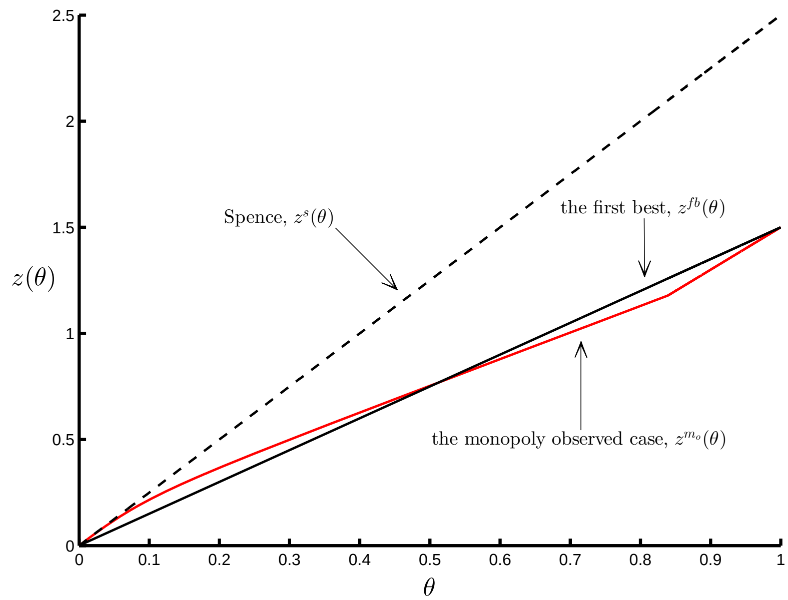

Figure 3: **Over-Education at the Low End.**

less than the first best. This is illustrated in Figure 3 which assumes the same numerical example as in Figure 2. To summarize, we have the following proposition:

Proposition 2. *Suppose that $\gamma > 1$, then for sufficiently large k, if the market contains both the partially covered and fully covered range, then there exits a cutoff $\tilde{\theta} \in (0, \theta_1^{m_o})$, such that $z^{m_o}(\theta) > z^{fb}(\theta)$ on $(0, \tilde{\theta})$, whereas $z^{m_o}(\theta) < z^{fb}(\theta)$ on $(\tilde{\theta}, 1)$.*

As is depicted by Figure 3, $z^{m_o}(\theta)$ is single-crossing $z^{fb}(\theta)$ from above in the interior of the partially covered range. The intuition for over-education occurring at the low end is that when $\gamma > 1$, signaling is relatively intense; if the transportation cost is relatively high, then to gain market share, the monopolist charges low prices, especially at the low end of the vertical market. As is illustrated in panel (b) of Figure 2, the tuition scheme is flat and close to 0 for low education levels. Thus, at the low end, signaling outweighs screening, leading to over-education. In addition, from Figure 3, $z^{m_o}(\theta)$ is bounded above by $z^s(\theta)$ which is the equilibrium education function in Spence's model. Intuitively, since tuition is fixed at zero, education is the least costly in Spence's signaling game, compared to other models; thus, the worker obtains the highest education level in Spence's model.

We are interested in the impacts of the unit transportation cost k on the equilibrium allocation, in particular, on the (vertical) market coverage and quantity distortion. As in Yang and Ye (2008), k measures the market's horizontal differentiation: a larger k means that the two schools are more horizontally differentiated. Following immediately from the previous analysis, Corollary 1 below shows that when the fully covered range exists and the vertical market is not fully covered, as horizontal differentiation increases, the monopolist raises the market coverage, offers more rent, and reduces the downward distortion in education level. When horizontal differentiation is eliminated, i.e., $k = 0$, the equilibrium outcome coincides with that of the observed case of Lu (2018). Formally, we have:

Corollary 1. *In the observed case under monopoly, when k is such that $\theta_0^{m_o} > 0$, $\theta_1^{m_o} < 1$, as k increases: (i) $z^{m_o}(\theta)$ increases on $(\theta_0^{m_o}, \theta_1^{m_o}]$ but remains the same on $(\theta_1^{m_o}, 1]$; (iii) $V^{m_o}(\theta)$ increases on $(\theta_0^{m_o}, 1]$; (ii) the market coverage $[\theta_0^{m_o}, 1]$ extends, whereas the fully covered range $[\theta_1^{m_o}, 1]$ shrinks. If $k = 0$, then the equilibrium outcome coincides with that of the observed case of Lu (2018).*

Intuitively, as horizontal differentiation rises, to maintain market share in the partially covered range, the monopolist has to provide the worker with more rent, which, according to Lemma 1, can be achieved by either increasing the market coverage, i.e., reducing θ_0, or supplying more education to those participated. The optimal allocation requires a balance between these two approaches. Corollary 1 shows that both methods will be employed in equilibrium when k increases. Corollary 1 also states that as k increases, the fully covered range shrinks. This is because the switching type θ_1's education level is pinned down by the IC constraint in the fully covered range, which does not directly depend on k; as education levels increase in the partially covered range, θ_1 must be higher accordingly. In other words, as the fixed fee of attending school kd_i increases, the marginal type θ_1 must be higher.

3.2 The Unobserved Case

We now turn to the unobserved case. Since we consider a symmetric equilibrium, we assume that the labor market offers the same wage to both schools' student for a given education level, thereby allowing us to drop the subscript of the wage schedule. Then, given some wage schedule $W(z)$, the monopolist solves:

$$\max \underbrace{\int_{\theta_0}^{\theta_1} [W(z(\theta)) - C(z(\theta), \theta) - V(\theta)] \frac{V(\theta)}{k} d\theta}_{\text{Phase I: partially covered range}} + \underbrace{\int_{\theta_1}^{1} [W(z(\theta)) - C(z(\theta), \theta) - V(\theta)] \frac{1}{4} d\theta}_{\text{Phase II: fully covered range}}$$

$$s.t.\ V'(\theta) = z(\theta),\ z'(\theta) \geq 0,\ V(\theta_1) = \frac{k}{4}.$$

If further $\theta_0 \in (0, 1]$, then we have $V(\theta_0) = 0$; otherwise, as in the observed case, we have to choose $V(\theta_0)$ optimally.

Similarly to the observed case, we define the *Hamiltonian* of the two phases as follows:

$$H_1(z, V, \lambda, \theta) = [W(z) - C(z, \theta) - V]\frac{V}{k} + \lambda z,$$
$$H_2(z, V, \lambda, \theta) = [W(z) - C(z, \theta) - V]\frac{1}{4} + \lambda z,$$

The key difference from the observed case is that here $W(z)$ is endogenously determined by the equilibrium condition. Suppose that the school-optimal separating equilibrium exits, and let $\langle z^*(\theta), V^*(\theta)\rangle$ solves the monopolist's problem, then from the Maximum Principle, we have the following first oder conditions (F.O.C.):

$$\frac{\partial}{\partial z}H_1(z^*(\theta), V^*(\theta), \lambda(\theta), \theta) = [W'(z^*(\theta)) - C_z(z^*(\theta), \theta)]\frac{V^*(\theta)}{k} + \lambda(\theta) = 0,$$
$$\frac{\partial}{\partial z}H_2(z^*(\theta), V^*(\theta), \lambda(\theta), \theta) = [W'(z^*(\theta)) - C_z(z^*(\theta), \theta)]\frac{1}{4} + \lambda(\theta) = 0,$$

along with the evolution rule for λ:

$$\dot{\lambda}(\theta) = -\frac{\partial}{\partial V}H_i\left(z^*(\theta), V^*(\theta), \lambda(\theta), \theta\right),\ i = 1, 2,$$

and the transversality condition $\lambda(1) = 0$.

Moreover, in equilibrium, the market belief should be correct: $W(z^*(\theta)) = Q(z^*(\theta), \theta)$. Thus, we have $W'(z^*(\theta)) = Q_z(z^*(\theta), \theta) + Q_\theta(z^*(\theta), \theta) \cdot \theta^{*\prime}(z)$. Combining these conditions and substituting the model assumptions, we derive an autonomous ODE for Phase I:

$$(2\gamma + 3)V - \frac{\gamma V\dot{V}\dddot{V}}{\ddot{V}^2} - 2\ddot{V}V + \frac{\gamma\dot{V}^2}{\ddot{V}} - \dot{V}^2 = 0. \tag{3.3}$$

To solve (3.3), we first consider the case in which the vertical market is partially covered, i.e., $\theta_0^* \in (0, 1]$, and thus, $V(\theta_0^*) = 0$. Given this boundary condition, it can be verified that the solution to (3.3) is given by

$$V^*(\theta) = \frac{4\gamma + 3}{8}(\theta - \theta_0^*)^2,\ z^*(\theta) = \frac{4\gamma + 3}{4}(\theta - \theta_0^*),$$

and thus, the wage schedule in Phase I is given by

$$W^*(z) = \frac{4\gamma}{4\gamma + 3}z^2 + (\gamma\theta_0^* + 1)z,$$

where the lowest participating type θ_0^* remains to be determined.

Then, we consider Phase II. Since $\lambda'(\theta) = \frac{1}{4}$ and $\lambda(1) = 0$, we have $\lambda(\theta) = \frac{\theta - 1}{4}$ in Phase II. Substituting $\lambda(\theta)$ and the condition $W(z) = Q(z, \theta(z))$ into the F.O.C. for z in Phase II, we obtain the following ODE:

$$W'(z) = 2(z - \frac{W}{\gamma z} + \frac{\gamma + 1}{\gamma}).$$

The general solution to this ODE is given by

$$W(z)=\frac{\gamma}{\gamma+1}z^2+\frac{2(\gamma+1)}{\gamma+2}z+c\cdot z^{\frac{\gamma}{2}},$$

where c is some parameter. To fully characterize $W(z)$, we need to pin down c. As is argued previously, the current model converges to Lu (2018) as $k\to 0$. Thus, we apply Lu (2018, Poposition 5.1) to the current model, assuming that $k=0$, and conclude that $c=0$. As such, we have fully characterized $W(z)$ for Phase II.

It thus follows that in Phase II, $V^*(\theta)$ and $z^*(\theta)$ are given by

$$V^*(\theta)=\frac{\gamma+1}{2}\theta^2-\frac{\gamma+1}{\gamma+2}\theta+\beta(\theta_1^*),\ z^*(\theta)=(\gamma+1)(\theta-\frac{1}{\gamma+2})$$

where $\beta(\theta_1^*)$ depends on the optimal switching type θ_1^* that remains to be determined. Then, by smooth pasting and the condition $V(\theta_1^*)=\frac{k}{4}$, θ_1^* and θ_0^* are given by

$$\theta_0^*=\frac{1}{\gamma+2}-\frac{\sqrt{2(4\gamma+3)k}}{4(\gamma+1)(4\gamma+3)},\ \theta_1^*=\frac{1}{\gamma+2}+\frac{\sqrt{2(4\gamma+3)k}}{4(\gamma+1)}. \tag{3.4}$$

Substituting θ_1^*, we have that in Phase II, $V(\theta)$ is given by

$$V^*(\theta)=\frac{k}{4}+(\theta-\theta_1^*)\left[\frac{(\gamma+1)(\theta+\theta_1^*)}{4}-\frac{\gamma+1}{\gamma+2}\right].$$

In addition, from (3.4), if $\frac{8(\gamma+1)^4}{(4\gamma+3)(\gamma+2)^2}<k<\frac{8(\gamma+1)^2(\gamma+3)^2}{(4\gamma+3)(\gamma+2)^2}$, then $\theta_0^*>0$ and $\theta_1^*>1$; thus, Phase I exits, while Phase II is never entered. In this case, θ_0^* is pinned down by the transversality condition $\lambda(1)=0$, such that $\theta_0^*=\frac{1}{2\gamma+3}>0$. Thus, for any $k>0$, we have $\theta_0^*>0$, that is, the vertical market is always partially covered.

Since $z^*(\theta)$ is increasing in both Phase I and II, and the initial condition is optimally chosen, we obtain the school-optimal separating equilibrium. Let $\langle z^{m_u}(\theta),V^{m_u}(\theta)\rangle$ be the equilibrium contract in the unobserved case under monopoly, and $\theta_0^{m_u}$ and $\theta_1^{m_u}$ be the lowest participating type and switching type, respectively. We summarize the equilibrium education allocation of the unobserved case in the proposition below.

Proposition 3. *In the unobserved case under monopoly, the symmetric school-optimal separating equilibrium exists. Specifically, for* $k\in\left(0,\frac{8(\gamma+1)^4}{(4\gamma+3)(\gamma+2)^2}\right)$,

$$z^{m_u}(\theta)=\begin{cases}\frac{4\gamma+3}{4}(\theta-\theta_0^{m_u}) & \text{if } \theta_0^{m_u}\le\theta<\theta_1^{m_u}\\ (\gamma+1)\theta-\frac{\gamma+1}{\gamma+2} & \text{if } \theta_1^{m_u}\le\theta\le 1,\end{cases}$$

where $\theta_0^{m_u}$ and $\theta_1^{m_u}$ are given by θ_0^* and θ_1^* in (3.4), respectively. For $k \geq \frac{8(\gamma+1)^4}{(4\gamma+3)(\gamma+2)^2}$,

$$z^{m_u}(\theta) = \frac{4\gamma+3}{4}(\theta - \theta_0^{m_u}), \ \text{if } \theta_0^{m_u} \leq \theta \leq 1,$$

where $\theta_0^{m_u} = \frac{1}{2\gamma+3}$. It follows that there exits a cutoff $\tilde{\theta} \in (\theta_0^{m_u}, 1)$, such that $z^{m_u}(\theta) < z^{fb}(\theta)$ on $(\theta_0^{m_u}, \tilde{\theta})$, whereas $z^{m_u}(\theta) > z^{fb}(\theta)$ on $(\tilde{\theta}, 1)$.

As is immediately implied by Proposition 3, the degree of horizontal differentiation that is measured by k has similar effects on education supply, the worker's gross utility and the market coverage as in the observed case. Specifically, we have the following corollary:

Corollary 2. *In the unobserved case under monopoly, when the market contains the fully covered range, as k increases: (i) $z^{m_u}(\theta)$ increases for $\theta \in (\theta_0^{m_u}, \theta_1^{m_u}]$ but remains the same for $\theta \in (\theta_1^{m_u}, 1]$. (ii) $V^{m_u}(\theta)$ increases for $\theta \in (\theta_0^{m_u}, 1]$; (iii) the market coverage $[\theta_0^{m_u}, 1]$ enlarges, whereas the fully covered range $[\theta_1^{m_u}, 1]$ shrinks. If $k = 0$, then the equilibrium outcome coincides with that of the unobserved case of Lu (2018).*

We are interested in the difference in equilibrium allocation between the observed and unobserved case. Proposition 3 shows that in contrast to the observed case, in the unobserved case, both under-education and over-education occur in equilibrium. Specifically, there exits a cutoff type such that all lower types obtain less education than the first-best, whereas the others obtain more than the first best. The next proposition shows further that in the unobserved case, the (vertical) market coverage is smaller than that of the observed case, whereas the fully covered range is larger in the unobserved case. Moreover, there exists a cutoff type in the partially covered range of the unobserved case, such that all lower types obtain less education in the unobserved case than in the observed case, whereas the others obtain more education in the unobserved case. Formally, we have:

Proposition 4. *For any $\gamma, k > 0$, we have $\theta_0^{m_u} > \theta_0^{m_o}$ and $\theta_1^{m_u} < \theta_1^{m_o}$. Furthermore, there exits a cutoff $\tilde{\theta} \in (\theta_0^{m_u}, \theta_1^{m_u})$, such that $z^{m_u}(\theta) < z^{m_o}(\theta)$ on $(\theta_0^{m_o}, \tilde{\theta})$, whereas $z^{m_u}(\theta) > z^{m_o}(\theta)$ on $(\tilde{\theta}, 1]$. The length of the interval $(\theta_0^{m_o}, \tilde{\theta})$ is increasing in k, and vanishes as $k \to 0$.*

Proposition 4 indicates that education levels are always higher in the unobserved case than in the observed case within the fully covered range of both cases. This result generalizes that of Lu (2018) in which the market contains only the fully covered range in both cases, and thus, the worker obtains more education in the unobserved case than in the observed case. As in Lu (2018), this result is driven by a *signal jamming effect.* Specifically, in the unobserved case, since the labor market cannot observe the actual cost of education, it does not know whether a difference in education level is caused by a tuition change or worker cost heterogeneity. Suppose that the monopolist lowers tuition so that the worker obtains more education than in the initial state. When the labor market observes the tuition scheme, it cuts wages, since any education level now corresponds to a lower-ability worker. This dampens the worker's demand for additional education. In contrast, when the labor market does not observe the tuition scheme, it does not adjust wages despite that tuition changes. Thus, the demand for education is more elastic in the unobserved case. This provides the monopolist with an incentive to secretly supply more education and persuades the labor market that the worker is more productive than is actually the case. Since in the observed case efficiency occurs at the top, over-education must occur at the high end in the unobserved case, as is predicted by Proposition 3. In equilibrium, however, the labor market correctly anticipates the monopolist's incentive and offers lower wages, as education is inflated. This reduces the worker's willingness to pay, and thus, the monopolist achieves lower profits.

Furthermore, Proposition 4 reveals a significant distinction between Lu (2018) and the current model. That is, an interval of vertical types at the low end obtain more education in the observed case than in the unobserved case; the length of this interval is increasing in the degree of horizontal differentiation and vanishes as the degree approaches zero. Intuitively, when horizontal differentiation rises, to maintain the market share in the partially covered range, the monopolist offers the worker higher rent by increasing both the market coverage and education levels. However, the increase in education supply is smaller in the unobserved case than in the observed case, especially at the low end of the market. This is because if the monopolist allocates the same education level to lower types as in the observed case, then the monopolist should allocate more education and leave more rent to higher types to remain incentive compatibility. But since those higher types already obtain higher education levels than in the observed case, supplying more education to them is not profitable. Thus, an interval of lower vertical types obtain less education in the unobserved case than in the observed case. As horizontal differentiation increases, this interval extends, meaning that the market coverage is larger in the observed case at any degree of horizontal differentiation. Indeed, Propositions 3 and 4 imply that the education function is uniformly steeper in the unobserved case than in the observed. These features are illustrated in Figure 4.

Proposition 4 implies that an interval of lower vertical types obtain lower gross utility in the unobserved case than in the observed case, whereas the others obtain higher gross utility in the unobserved case, and the length of this interval is increasing in the degree of horizontal differentiation. Similarly, the tuition scheme in the unobserved case T^{m_u} is higher

(a) $\gamma = 1$, $k = 0.5$ (b) $\gamma = 1$, $k = 1$

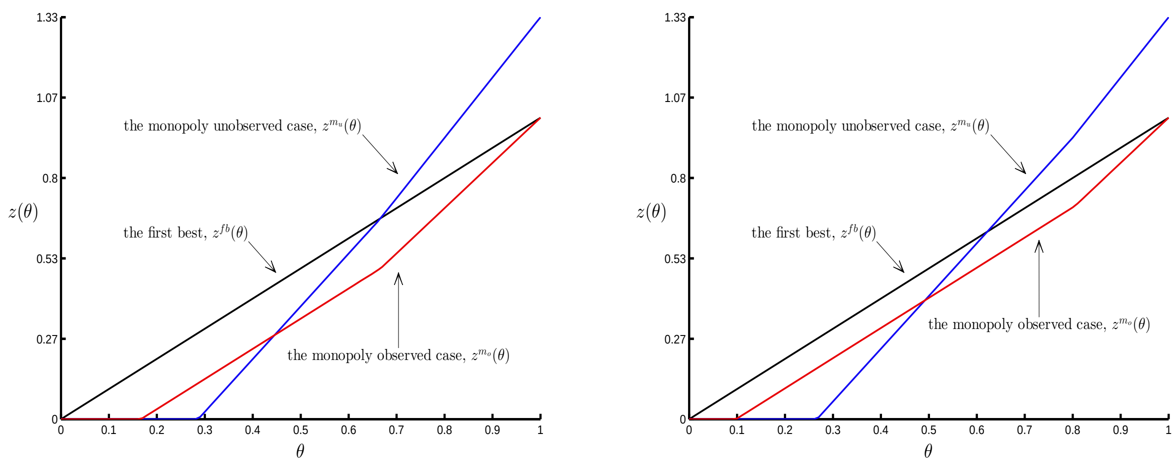

Figure 4: **Education Comparison between the Observed and Unobserved Case**

than that in the observed case T^{m_o} at the left tail of the common domain, such a region is also increasing in horizontal differentiation. These results differ from that of Lu (2018) in which tuition rates are uniformly lower and the worker is always better-off in the unobserved case than in the observed case. To summarize:

Proposition 5. *For any $\gamma, k > 0$, there exits a cutoff $\tilde{\theta} \in (\theta_0^{m_u}, 1)$, such that $V^{m_u}(\theta) < V^{m_o}(\theta)$ on $(\theta_0^{m_o}, \tilde{\theta})$, whereas $V^{m_u}(\theta) > V^{m_o}(\theta)$ on $(\tilde{\theta}, 1]$. The length of the interval $(\theta_0^{m_o}, \tilde{\theta})$ is increasing in k, and vanishes as $k \to 0$. Furthermore, there exists a cutoff $\tilde{z} \in (0, 1)$, such that $T^{m_u}(z) > T^{m_o}(z)$ on $(0, \tilde{z})$, whereas $T^{m_u}(z) > T^{m_o}(z)$ on $(\tilde{z}, 1]$. The length of the interval $(0, \tilde{z})$ is increasing in k, and vanishes as $k \to 0$.*

From Corollaries 1 and 2, we have that the worker's gross utility is increasing in the degree of horizontal differentiation in both the observed and unobserved case. Thus, if the worker is close to either school, i.e., $\min\{d_1, d_2\}$ is small enough, then his net utility is also increasing in the degree of horizontal differentiation. Intuitively, as horizontal differentiation increases, the worker's value for education becomes more dispersed, which corresponds to a clockwise rotation in demand (Johnson and Myatt 2006). Consequently, the monopolist lowers prices as the marginal consumer's willingness to pay is lower. This benefits those infra-marginal consumers who are close to either school. Proposition 5 implies that a low-ability worker who is close to either school benefits more from the increase in horizontal differentiation in the observed case than in the unobserved case.

4 Duopoly

In this section, we consider a duopoly education market in which each school chooses a contract to maximize its expected profit, given the other school's contract. The purpose of this section is to investigate the effects of market competition on education supply and the market coverage, compared to the monopoly benchmark, for the observed and unobserved case separately. For ease of comparison, we focus on the case in which the vertical market is partially covered in the monopoly case. As such, we assume throughout this section that $k < \bar{k} := \min\left\{\frac{2(\gamma+1)^2}{\gamma+3}, \frac{2(\gamma+3)}{(\gamma+1)^2}\right\}$. As in the monopoly case, we consider symmetric equilibrium. We start our analysis with the observed case.

4.1 The Observed Case

Suppose that a symmetric equilibrium exits, such that both schools choose an identical contract $\langle z^*(\theta), V^*(\theta)\rangle$. Thus, given that the other school chooses $\langle z^*(\theta), V^*(\theta)\rangle$, school i's best response is to chooses $\langle z_i(\theta), V_i(\theta)\rangle = \langle z^*(\theta), V^*(\theta)\rangle$. Given its expected profit in (2.2), school i's problem can be stated as

$$\max \underbrace{\int_{\theta_{0_i}}^{\theta_1} [S(z_i(\theta),\theta) - V_i(\theta)] \frac{V_i(\theta)}{k} d\theta}_{\text{Phase I: the local monopoly range}} + \underbrace{\int_{\theta_1}^{1} [S(z_i(\theta),\theta) - V_i(\theta)] \cdot \left[\frac{1}{4} + \frac{V_i(\theta) - V^*(\theta)}{2k}\right] d\theta}_{\text{Phase II: the competition range}}.$$

$$s.t.\ V_i'(\theta) = z(\theta),\ z_i'(\theta) \geq 0,\ V_i(\theta_1) + V^*(\theta_1) = \frac{k}{2}.$$

If further $\theta_{0_i} \in (0,1]$, then we have $V_i(\theta_{0_i}) = 0$; otherwise, $V_i(\theta_{0_i})$ is free. Analogously, we define the Hamiltonian of the two phases and substitute $S(z,\theta)$, then we have:

$$H_1(z_i, V_i, \lambda, \theta) = [(\gamma+1)\theta z_i - z_i^2 - V_i]\frac{V_i}{k} + \lambda z_i,$$
$$H_2(z_i, V_i, \lambda, \theta) = [(\gamma+1)\theta z_i - z_i^2 - V_i] \cdot \left(\frac{1}{4} + \frac{V_i - V^*}{2k}\right) + \lambda z_i,$$

Note that Phase I is exactly the same as that in the monopoly case. If $\theta_{0_i} \in (0,1]$, then the solution to Phase I is also given by:

$$V^*(\theta) = \frac{\gamma+3}{8}(\theta - \theta_0^*)^2,\ z^*(\theta) = \frac{\gamma+3}{4}(\theta - \theta_0^*).$$

Then, we consider Phase II. By the Maximum Principle, we obtain the necessary conditions:

$$0 = \left[(\gamma+1)\theta z^*(\theta) - z^{*2}(\theta)\right] \cdot \frac{1}{4} + \lambda(\theta),$$
$$\dot{\lambda}(\theta) = \frac{1}{4} - \frac{(\gamma+1)\theta z^*(\theta) - z^{*2}(\theta) - V^*(\theta)}{2k},$$

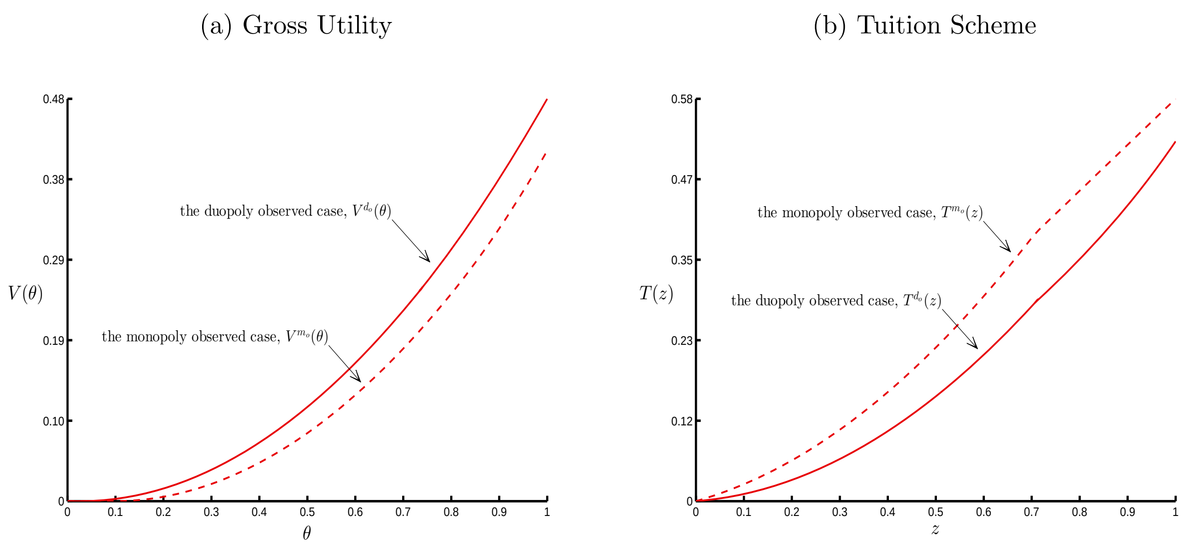

Figure 5: **A Numerical Solution.** This figure assumes that $\gamma = 1$ and $k = 1$.

combined with the transversality condition $\lambda(1) = 0$. Eliminating $\lambda(\theta)$ from the above two equations, we obtain the following ODE:

$$\ddot{V}^* = \frac{\gamma+2}{2} - \frac{1}{k}[(\gamma+1)\theta\dot{V}^* - \dot{V}^{*2} - V^*].$$

In equilibrium, $V_i(\theta) = V^*(\theta)$, and thus, $V^*(\theta_1) = \frac{k}{4}$. From smooth pasting and the solution to Phase I, we have $\dot{V}^*(\theta_1) = z^*(\theta_1) = \frac{\sqrt{2(\gamma+3)k}}{4}$. In addition, $\lambda(1) = 0$ implies that $\dot{V}^*(1) = z^*(1) = \frac{\gamma+1}{2}$. Thus, the existence of equilibrium reduces to the existence of $\theta_1 \in (0, 1]$ and the existence of a convex solution $V(\theta)$ over $[\theta_1, 1]$, satisfying:

$$\ddot{V} = \frac{\gamma+2}{2} - \frac{1}{k}[(\gamma+1)\theta\dot{V} - \dot{V}^2 - V] \tag{4.1}$$

$$s.t.\ V(\theta_1) = \frac{k}{4},\ \dot{V}(\theta_1) = \frac{\sqrt{2(\gamma+3)k}}{4},\ \dot{V}(1) = \frac{\gamma+1}{2}.$$

Note that (4.1) is not a standard BVP, as it involves an endogenous endpoint θ_1. As far as we know, no existing BVP theorem can be applied directly to show the existence and uniqueness of the solution to this problem. The oder-reduce techniques introduced by Rochet and Stole (2002) and Yang and Ye (2008) cannot be applied to (4.1) either. In this regard, we solve program (4.1) using numerical methods. Let $\langle z^{d_o}(\theta), V^{d_o}(\theta)\rangle$ be the equilibrium contract in the observed case under duopoly, and $\theta_0^{d_o}$ and $\theta_1^{d_o}$ be the lowest participating type and switching type, respectively. In Figure 5, panel (a) depicts a convex solution $V^{d_o}(\theta)$, along with the worker's gross utility in the monopoly observed case $V^{m_o}(\theta)$; panel (b) depicts the associated equilibrium tuition scheme $T^{d_o}(z)$, along with that of the monopoly observed case $T^{m_o}(z)$. It turns out that under duopoly, tuition is lower and the worker obtains higher utility than under monopoly. We now summarize the equilibrium education allocation in the proposition below.

Proposition 6. *Suppose that $k \in (0, \bar{k})$, then in the observed case under duopoly, the symmetric school-optimal separating equilibrium exists, such that*

$$z^{d_o}(\theta) = \begin{cases} \frac{\gamma+3}{4}(\theta - \theta_0^{d_o}) & \text{if } \theta_0^{d_o} \le \theta < \theta_1^{d_o} \\ \dot{V}^{d_o}(\theta) & \text{if } \theta_1^{d_o} \le \theta \le 1, \end{cases}$$

where $V^{d_o}(\theta)$ and $\theta_1^{d_o}$ are the solution to problem (4.1), and $\theta_0^{d_o} = \theta_1^{d_o} - \sqrt{\frac{2k}{\gamma+3}}$.

Proposition 6 indicates that under duopoly, the equilibrium is discontinuous at $k = 0$. When $k = 0$, the equilibrium is a Bertrand-Spence equilibrium in which tuition is pushed down to 0 due to a symmetric Bertrand competition, and thus, the market is fully covered and the education selection is predicted by Spence's model. However, since social surplus is close to 0 for sufficiently low types, so long as $k > 0$, each school becomes a local monopoly for those types. Thus, each school finds it profitable to exclude some very low types from education. Since the threshold is endogenously determined, it leads to distortion for inframarginal types. In contrast, in Armstrong and Vickers (2001) and Rochet and Stole (2002), the lowest type can generate sufficiently high social surplus, thus, when the market is fully covered, both competing duopolists offer a cost-plus-fee tariff. This is because given that the competitor chooses such a pricing strategy, each duopolist finds it more profitable to make an efficient offer with a higher fixed fee than any inefficient offer.

Going forward, we investigate the impacts of market competition on education supply and the market coverage. The next proposition shows that in contrast to the monopoly case, both education supply and the market coverage are higher under duopoly. This is illustrated in Figure 6. A similar result has been obtained by Yang and Ye (2008).

Proposition 7. *Given $k \in (0, \bar{k})$, we have $\theta_0^{d_o} < \theta_0^{m_o}$ and $z^{d_o}(\theta) > z^{m_o}(\theta)$ for $\theta \in (\theta_0^{d_o}, 1)$. It follows that in contrast to the monopoly case, more worker types (in terms of both vertical and horizontal type) receive education, and each participating type obtains higher net utility.*

Intuitively, under duopoly, the two schools compete with each other in the fully covered range by providing the worker with more rent than in the monopoly case, thereby extending the fully covered range. Moreover, this relaxes the IC constraint. Specifically, each school fears less about allocating more education to lower types thereby providing higher types with

(a) Education Supply (b) Market Coverage

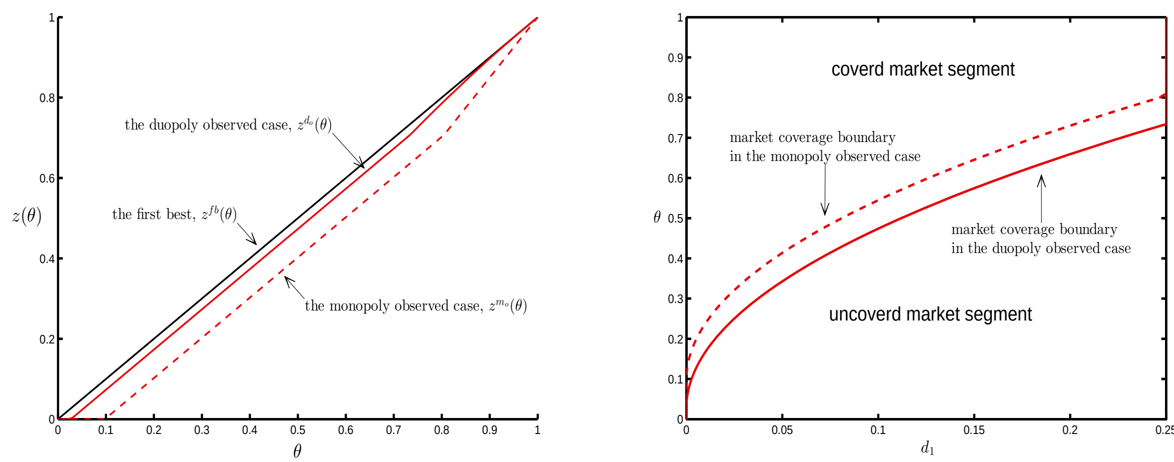

Figure 6: **The Impacts of Market Competition.** This figure assumes that $\gamma = 1$ and $k = 1$.

more rent, as higher types will enjoy more rent anyway due to market competition. Hence, the schools increase education supply for all participating types, and include some of those who are not served in the monopoly case.

4.2 The Unobserved Case

We now turn to the unobserved case. Suppose that a symmetric equilibrium exits, in which both schools choose an identical contract $\langle z^*(\theta), V^*(\theta)\rangle$, and the labor market offers the same wage schedule $W^*(z)$ for both schools' student. Given the wage schedule $W^*(z)$ and that the other school chooses $\langle z^*(\theta), V^*(\theta)\rangle$, the school's problem can be stated as

$$\max \underbrace{\int_{\theta_{0_i}}^{\theta_1} [W^*(z_i(\theta)) - C(z_i(\theta),\theta) - V_i(\theta)]\,\frac{V_i(\theta)}{k}d\theta}_{\text{Phase I: the local monopoly range}}$$

$$+\underbrace{\int_{\theta_1}^{1} [W^*(z_i(\theta)) - C(z_i(\theta),\theta) - V_i(\theta)] \cdot \left[\frac{1}{4} + \frac{V_i(\theta) - V^*(\theta)}{2k}\right] d\theta}_{\text{Phase II: the competition range}}\,.$$

$$s.t.\ V_i'(\theta) = z(\theta),\ \ z_i'(\theta) \geq 0,\ \ V_i(\theta_1) + V^*(\theta_1) = \frac{k}{2}.$$

If further $\theta_{0_i} \in (0,1]$, then we have $V_i(\theta_{0_i}) = 0$; otherwise, $V_i(\theta_{0_i})$ is free. Analogously, we define the Hamiltonian of the two phases and substitute $S(z,\theta)$, then we have:

$$H_1(z_i, V_i, \lambda, \theta) = [W^*(z_i) - C(z_i,\theta) - V_i]\frac{V_i}{k} + \lambda z_i,$$
$$H_2(z_i, V_i, \lambda, \theta) = [W^*(z_i) - C(z_i,\theta) - V_i] \cdot \left(\frac{1}{4} + \frac{V_i - V^*}{2k}\right) + \lambda z_i,$$

As in the observed case, Phase I coincides with that in the monopoly case. If $\theta_{0_i} \in (0,1]$, then the solution to Phase I is also given by:

$$V^*(\theta) = \frac{4\gamma+3}{8}(\theta - \theta_0^*)^2,\ z^*(\theta) = \frac{4\gamma+3}{4}(\theta - \theta_0^*).$$

Then, we consider Phase II. By the Maximum Principle, we obtain the necessary conditions:

$$0 = \left[W^{*'}(z^*(\theta)) - 2z^*(\theta) - 1 + \theta\right] \cdot \frac{1}{4} + \lambda(\theta),$$
$$\dot{\lambda}(\theta) = \frac{1}{4} - \frac{W^*(z^*(\theta)) - z^{*2}(\theta) - (1-\theta)z^*(\theta) - V^*(\theta)}{2k},$$

combined with the transversality condition $\lambda(1) = 0$. Moreover, the market belief correctness means that $W^*(z) = Q(z, \theta^*(z))$. Then, substituting $W^*(z)$ and eliminating λ from the above two equations, we obtain the following ODE:

$$\dddot{V}^* = \frac{(\gamma+2)\ddot{V}^* + (\gamma-2)\ddot{V}^{*2}}{\gamma\dot{V}^*} + \frac{2}{\gamma k}\left[\frac{V^*\ddot{V}^*}{\dot{V}^*} - (\gamma+1)\theta\dot{V}^* + \dot{V}^*\ddot{V}^*\right].$$

In equilibrium, $V_i(\theta) = V^*(\theta)$, and thus, $V^*(\theta_1) = \frac{k}{4}$. From smooth pasting and the solution to Phase I, we have $\dot{V}^*(\theta_1) = z^*(\theta_1) = \frac{\sqrt{2(4\gamma+3)k}}{4}$. In addition, $\lambda(1) = 0$ combined with the F.O.C. for z implies that $W^{*'}(z(1)) - 2z(1) = 0$, meaning that $\frac{[\gamma - 2\ddot{V}^*(1)]\dot{V}^*(1)}{\ddot{V}^*(1)} = \gamma + 1$. Thus, the existence of equilibrium reduces to the existence of $\theta_1 \in (0,1]$ and the existence of a convex solution $V(\theta)$ over $[\theta_1, 1]$, satisfying:

$$\dddot{V} = \frac{(\gamma+2)\ddot{V} + (\gamma-2)\ddot{V}^2}{\gamma\dot{V}} + \frac{2}{\gamma k}\left[\frac{V\ddot{V}}{\dot{V}} - (\gamma+1)\theta\dot{V} + \dot{V}\ddot{V}\right] \tag{4.2}$$

$$s.t.\ V(\theta_1) = \frac{k}{4},\ \dot{V}(\theta_1) = \frac{\sqrt{2(4\gamma+3)k}}{4},\ \frac{[\gamma - 2\ddot{V}(1)]\dot{V}(1)}{\ddot{V}(1)} = \gamma + 1.$$

Clearly, there is no closed form solution to (4.2) in general. Thus, we solve program (4.2) using numerical methods. Let $\langle z^{d_u}(\theta), V^{d_u}(\theta)\rangle$ be the equilibrium contract in the unobserved case under duopoly, and $\theta_0^{d_u}$ and $\theta_1^{d_u}$ be the lowest participating type and switching type, respectively. In Figure 7, panel (a) depicts a convex solution $V^{d_u}(\theta)$, along with the worker's

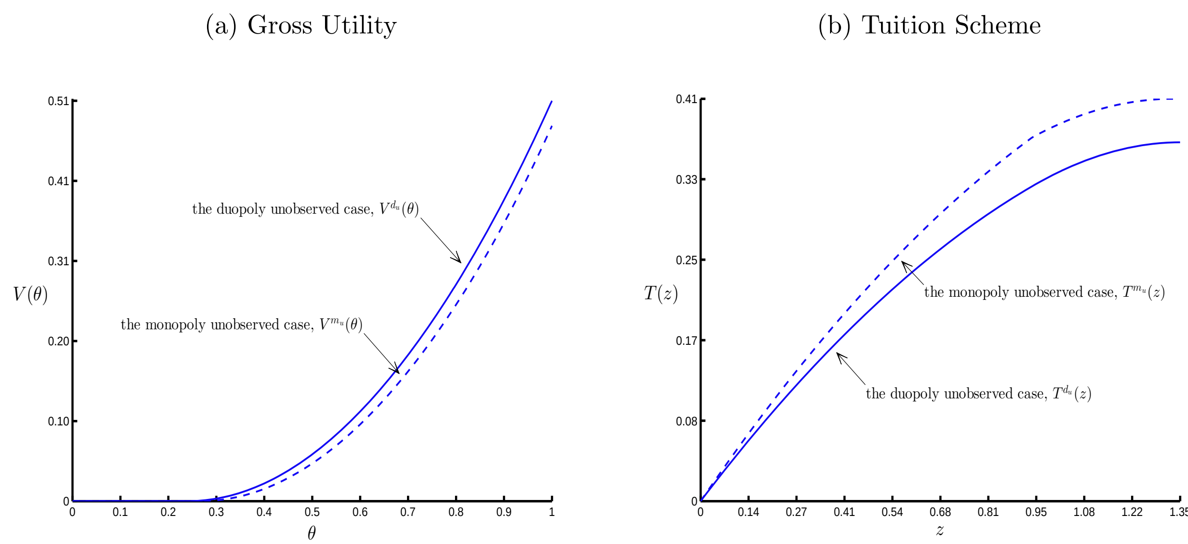

Figure 7: **A Numerical Solution.** This figure assumes that $\gamma = 1$ and $k = 1$.

gross utility in the monopoly observed case $V^{m_u}(\theta)$; panel (b) depicts the associated equilibrium tuition scheme $T^{d_u}(z)$, along with that of the monopoly observed case $T^{m_u}(z)$. As in the observed case, under duopoly, tuition is lower and the worker obtains higher utility than under monopoly. We now summarize the equilibrium education allocation in the following.

Proposition 8. *Suppose that $k \in (0, \bar{k})$, then in the unobserved case under duopoly, the symmetric school-optimal separating equilibrium exists, such that*

$$z^{d_u}(\theta) = \begin{cases} \frac{4\gamma+3}{4}(\theta - \theta_0^{d_u}) & \text{if } \theta_0^{d_u} \leq \theta < \theta_1^{d_u} \\ \dot{V}^{d_u}(\theta) & \text{if } \theta_1^{d_u} \leq \theta \leq 1, \end{cases}$$

where $V^{d_u}(\theta)$ and $\theta_1^{d_u}$ are the solution to problem (4.2), and $\theta_0^{d_u} = \theta_1^{d_u} - \sqrt{\frac{2k}{4\gamma+3}}$.

As in the observed case, the equilibrium is discontinuous at $k = 0$ in the unobserved case too. This is because when $k = 0$, the equilibrium is a Bertrand-Spence equilibrium as in the observed case. But so long as $k > 0$, both schools become a local monopoly for sufficiently low types. Consequently, both schools find it profitable to exclude some very low types and thus induce distortion for infra-marginal types.

In addition, we are interested in the impacts of market competition on education supply and the market coverage in the unobserved case. Unfortunately, we cannot obtain a clear result rigorously. This is mainly due to that in the unobserved case under both monopoly and duopoly, the highest type's education level is not fixed at the first-best but is determined endogenously in equilibrium. Thus, we cannot use the method in the proof of Proposition 7,

(a) Education Supply

(b) Market Coverage

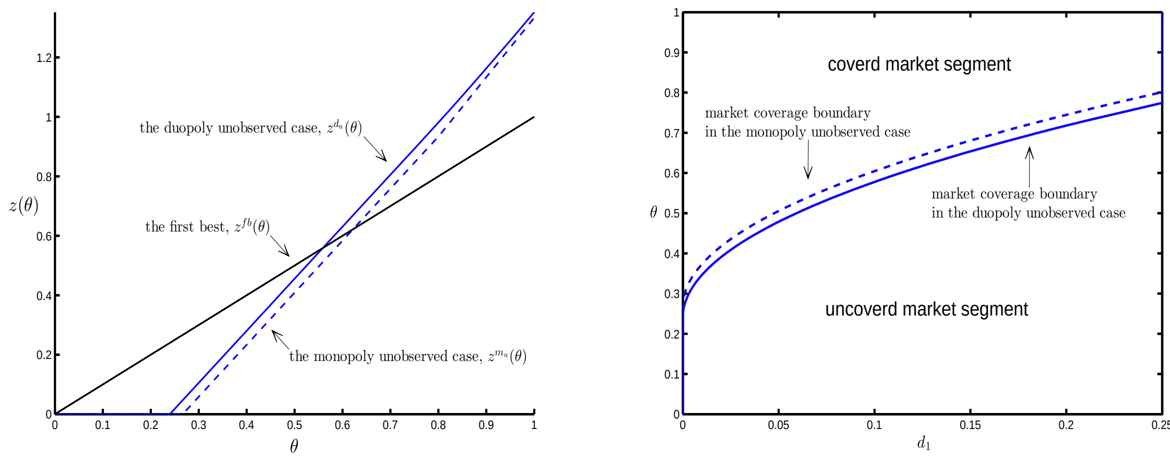

Figure 8: **The Impacts of Market Competition.** This figure assumes that $\gamma = 1$ and $k = 1$.

and we are unable to derive any result from the corresponding ODEs either. In Figure 8, we illustrate a numerical example assuming that $\gamma = 1$ and $k = 1$. It turns out that education supply and the market coverage are indeed higher under duopoly than under monopoly.

The intuition of Figure 8 deserves some comments. Suppose that both schools retain the contract of the monopoly case, and hence, the labor market offers the same wage schedule. Then, given the other's contract, each school has an incentive to supply more education. The reason is twofold. First, as in the observed case, each school has an incentive to supply more education to steal the market share from the other in the competition range. Second, since the labor market does not observe the actual tuition scheme, supplying more education can induce the labor market to regard the worker as having higher ability, thereby increasing the worker's willingness to pay. Thus, both schools will rise education supply in the competition range. This in turn relaxes the IC constraint for lower types. Since the signal jamming effect also exists in the local monopoly range, each school will supply more education in this range, and will also include some of those who are not served under monopoly. A noticeable feature of Figure 8 is that the increase in education supply is relatively small at the high end of the market. A possible intuition is that the schools have already allocated much education, compared to that in the observed case, to these types under monopoly. Thus, the schools find it less profitable to allocate more education to those high types.

5 Conclusion

In this paper, we study nonlinear pricing for horizontally differentiated products that provide signaling value to consumers, who choose how much to purchase as a signal to the receivers. We characterized the optimal symmetric price schedules under different market structures. The equilibrium depends critically on whether the signal receivers observe the sellers' price schedules, as well as on the market structure. Under monopoly, when the receivers observe the price schedule, the market is partially covered, and quantity is downward distorted if there is slight horizontal differentiation. As the degree of horizontal differentiation rises, the market coverage rises, and the downward distortion decreases. When the degree is sufficiently high, for some intermediate level of signaling intensity, the monopolistic allocation achieves the first-best; for higher signaling intensities, quantity is upward distorted at the low end. In contrast, when the receivers do not observe the price schedule, the market is always partially covered, and the allocation is more dispersed than that in the observed case. When the market structure changes from monopoly to duopoly, market competition results in a higher market coverage and larger quantities for both the observed and unobserved case.

By analyzing the products that provide signaling value to consumers who possess private information, our framework derives qualitatively different welfare implications from standard competitive nonlinear pricing models. In addition, our framework allows us to examine the interaction between horizontal competition and the transparency of pricing, and assess the joint effects of these two forces on the equilibrium allocation and welfare.

A Appendix

A.1 Omitted Proofs

Proof of Proposition 4.

Proof. We first prove that $\theta_0^{m_u} > \theta_0^{m_o}$. From (3.2) and (3.4), we have

$$\begin{aligned}\theta_0^{m_u} - \theta_0^{m_o} &= \frac{(\gamma+1)\sqrt{2(\gamma+3)k}}{2(\gamma+2)(\gamma+3)} - \frac{\sqrt{2(4\gamma+3)k}}{4(\gamma+1)(4\gamma+3)} \\ &= \frac{\gamma+1}{2(\gamma+2)}\sqrt{\frac{2k}{\gamma+3}}\left[1 - \frac{\gamma+2}{2(\gamma+1)^2}\sqrt{\frac{\gamma+3}{4\gamma+3}}\right]\end{aligned}$$

It can be easily verified that the value of the above bracket is positive for any $\gamma > 0$. Thus, we have $\theta_0^{m_u} > \theta_0^{m_o}$. Similarly, we have

$$\theta_1^{m_u} - \theta_1^{m_o} = \frac{\sqrt{2(4\gamma+3)k}}{4(\gamma+1)} \left[1 - \frac{2(\gamma+1)}{\gamma+2}\sqrt{\frac{\gamma+3}{4\gamma+3}}\right]$$

It can be easily verified that the value of the above bracket is negative for any $\gamma > 0$. Thus, we have $\theta_1^{m_u} < \theta_1^{m_o}$. This completes the proof of the first statement.

We then prove that there exits a cutoff $\tilde{\theta} \in (\theta_0^{m_u}, \theta_1^{m_u})$, such that $z^{m_u}(\theta) < z^{m_o}(\theta)$ on $(\theta_0^{m_o}, \tilde{\theta})$, but $z^{m_u}(\theta) > z^{m_o}(\theta)$ on $(\tilde{\theta}, 1]$. From Propositions 1 and 3, we have $z^{m_u}(1) > z^{m_o}(1)$. Since $z^{m_u}(\theta_0^{m_u}) = z^{m_o}(\theta_0^{m_o}) = 0$ and $\theta_0^{m_u} > \theta_0^{m_o}$, we have that $z^{m_u}(\theta)$ intersects $z^{m_o}(\theta)$ at least once. Let $\tilde{\theta}$ be one of the intersecting points. We prove that $\tilde{\theta} \in (\theta_0^{m_u}, \theta_1^{m_u})$. Suppose not, then we have $\tilde{\theta} \in [\theta_1^{m_u}, 1)$. We shall consider two cases: (i) $\tilde{\theta} \in [\theta_1^{m_u}, \theta_1^{m_o})$; (ii) $\tilde{\theta} \in (\theta_1^{m_o}, 1)$. Suppose that (i) holds, then we have

$$(\gamma+1)\tilde{\theta} - \frac{\gamma+1}{\gamma+2} = \frac{\gamma+3}{4}(\tilde{\theta} - \theta_0^{m_o}).$$

It follows that

$$\tilde{\theta} - \theta_0^{m_o} = \frac{4\gamma+4}{3\gamma+1}\left(\frac{1}{\gamma+2} - 1\right) < 0,$$

leading to a contradiction. Then, we consider (ii). If (ii) is true, then we have

$$(\gamma+1)\tilde{\theta} - \frac{\gamma+1}{\gamma+2} = \frac{(\gamma+2)\tilde{\theta}}{2} - \frac{1}{2}.$$

It follows that $\tilde{\theta} = \frac{1}{\gamma+2} < \theta_1^{m_o}$, leading to a contradiction. Thus, we have $\tilde{\theta} \in (\theta_0^{m_u}, \theta_1^{m_u})$. It remains to show that such a $\tilde{\theta}$ is unique. To see this, note that $z^{m_u'}(\theta) > z^{m_o'}(\theta)$ on $(\theta_0^{m_u}, 1)$, as $\tilde{\theta} \in (\theta_0^{m_u}, \theta_1^{m_u})$. Since $z^{m_u}(\tilde{\theta}) = z^{m_o}(\tilde{\theta})$, $z^{m_u}(\theta)$ is single-crossing $z^{m_o}(\theta)$ from below at $\tilde{\theta}$. This completes the proof of this statement. Finally, note that both $\theta_0^{m_o}$ and $\theta_1^{m_u}$ converge to $\frac{1}{\gamma+2}$ as $k \to 0$. Thus, $(\theta_0^{m_o}, \theta_1^{m_u})$ vanishes as $k \to 0$. The proposition is thus proven. □

Proof of Proposition 5.

Proof. We first prove that there exits a cutoff $\tilde{\theta} \in (\theta_0^{m_u}, 1)$, such that $V^{m_u}(\theta) < V^{m_o}(\theta)$ on $(\theta_0^{m_o}, \tilde{\theta})$, but $V^{m_u}(\theta) > V^{m_o}(\theta)$ on $(\tilde{\theta}, 1]$. From Lemma 1, we have

$$V^{m_u}(1) = \frac{k}{4} + \int_{\theta_1^{m_u}}^{1} z^{m_u'}(\theta)d\theta > \frac{k}{4} + \int_{\theta_1^{m_o}}^{1} z^{m_o'}(\theta)d\theta = V^{m_o}(1).$$

The inequality is due to that $\theta_1^{m_u} < \theta_1^{m_o}$ and $z^{m_u'}(\theta) > z^{m_o'}(\theta)$ on $(\theta_0^{m_u}, 1)$ according to the proof of Proposition 4. Since $V^{m_u}(\theta_0^{m_u}) = V^{m_o}(\theta_0^{m_o}) = 0$ and $\theta_0^{m_u} > \theta_0^{m_o}$, it follows from

Lemma 1 and the single-crossing between $z^{m_u}(\theta)$ and $z^{m_o}(\theta)$ that $V^{m_u}(\theta)$ is single-crossing $V^{m_o}(\theta)$ from below at some $\tilde{\theta} \in (\theta_0^{m_u}, 1)$. This completes the proof of this statement.

We then prove that there exists a cutoff $\tilde{z} \in (0, z^{m_o}(1))$, such that $T^{m_u}(z) > T^{m_o}(z)$ on $(0, \tilde{z})$, but $T^{m_u}(z) > T^{m_o}(z)$ on $(\tilde{z}, z^{m_o}(1)]$. Let $\tilde{\theta}'$ be the cutoff such that $z^{m_u}(\theta) < z^{m_o}(\theta)$ on $(\theta_0^{m_o}, \tilde{\theta}')$, but $z^{m_u}(\theta) > z^{m_o}(\theta)$ on $(\tilde{\theta}', 1]$. From the worker's F.O.C. in both cases, we have

$$W^{m_o'}(z) - T^{m_o'}(z) = C_z(z, \theta^{m_o}(z)) \ , W^{m_u'}(z) - T^{m_u'}(z) = C_z(z, \theta^{m_u}(z)).$$

Integrating both differential equations from 0 to $z^{m_o}(1)$, we have

$$W^{m_o}(z^{m_o}(1)) - T^{m_o}(z^{m_o}(1)) = C(z^{m_o}(1), 1),$$
$$W^{m_u}(z^{m_o}(1)) - T^{m_u}(z^{m_o}(1)) = C(z^{m_o}(1), \theta^{m_u}(z^{m_o}(1))).$$

From Proposition 4, we have $\theta^{m_u}(z^{m_o}(1)) < 1$. Since $W^{m_o}(z) = Q(z, \theta^{m_o}(z))$ and $W^{m_u}(z) = Q(z, \theta^{m_u}(z))$, $W^{m_o}(z^{m_o}(1)) > W^{m_u}(z^{m_o}(1))$. Also, since $C_{z\theta} < 0$, $C(z^{m_o}(1), \theta^{m_u}(z^{m_o}(1))) > C(z^{m_o}(1), 1)$. It follows that $T^{m_o}(z^{m_o}(1)) > T^{m_u}(z^{m_o}(1))$.

Moreover, since $z^{m_u}(\theta) < z^{m_o}(\theta)$ on $(\theta_0^{m_o}, \tilde{\theta}')$ and both $z^{m_u}(\theta)$ and $z^{m_o}(\theta)$ are increasing, $\theta^{m_u}(z) > \theta^{m_o}(z)$ on $(\theta_0^{m_o}, \tilde{\theta}')$. Thus, $C_z(z, \theta^{m_u}(z)) < C_z(z, \theta^{m_o}(z))$ on $(\theta_0^{m_o}, \tilde{\theta}')$. It follows that $W^{m_u}(z) - T^{m_u}(z) < W^{m_o}(z) - T^{m_o}(z)$ on $(0, z^{m_o}(\tilde{\theta}')]$. Since $W^{m_o}(z) = Q(z, \theta^{m_o}(z))$ and $W^{m_u}(z) = Q(z, \theta^{m_u}(z))$ on $(0, z^{m_o}(\tilde{\theta}')]$, $W^{m_u}(z) > W^{m_o}(z)$ on $(0, z^{m_o}(\tilde{\theta}')]$. Thus, it is readily confirmed that $T^{m_u}(z) > T^{m_o}(z)$ on $(0, z^{m_o}(\tilde{\theta}')]$. However, since $T^{m_o}(z^{m_o}(1)) > T^{m_u}(z^{m_o}(1))$, continuity implies that $T^{m_u}(z)$ must intersect $T^{m_o}(z)$ at some $\tilde{z} > z^{m_o}(\tilde{\theta}')$. It remains to prove that such $\tilde{z}$ is unique. To see this, note that for both the observed case

$$T^{m_o}(z) = S(z, \theta^{m_o}(z)) - V^{m_o}(\theta^{m_o}(z)),$$
$$T^{m_u}(z) = S(z, \theta^{m_u}(z)) - V^{m_u}(\theta^{m_u}(z)).$$

Differentiating both equations w.r.t. z and substituting, we have

$$T^{m_o'}(z) = S_z(z, \theta^{m_o}(z)) + S_\theta(z, \theta^{m_o}(z)) \cdot \theta^{m_o'}(z) - V^{m_o'}(\theta^{m_o}(z)) \cdot \theta^{m_o'}(z)$$
$$T^{m_u'}(z) = S_z(z, \theta^{m_u}(z)) + S_\theta(z, \theta^{m_u}(z)) \cdot \theta^{m_u'}(z) - V^{m_u'}(\theta^{m_u}(z)) \cdot \theta^{m_u'}(z).$$

Substituting $S(z, \theta)$ and note that $V'(\theta) = z(\theta)$, we have

$$T^{m_o'}(z) = (\gamma + 1)\theta^{m_o}(z) - 2z + \gamma z \cdot \theta^{m_o'}(z),$$
$$T^{m_u'}(z) = (\gamma + 1)\theta^{m_u}(z) - 2z + \gamma z \cdot \theta^{m_u'}(z).$$

Since $\tilde{z} > z^{m_o}(\tilde{\theta}')$, for any $z \in (\tilde{z}, z^{m_o}(1))$, we have $\theta^{m_o}(z) > \theta^{m_u}(z)$ by the definition of $\tilde{\theta}'$. From the proof of Proposition 4, we have $z^{m_u'}(\theta) > z^{m_o'}(\theta)$ on $(\theta_0^{m_u}, 1)$. This implies that $\theta^{m_o'}(z) > \theta^{m_u'}(z)$ on $(0, z^{m_o}(1))$. Thus, we have $T^{m_o'}(z) > T^{m_u'}(z)$ on $(\tilde{z}, z^{m_o}(1))$. Since $T^{m_o}(\tilde{z}) = T^{m_u}(\tilde{z})$, such $\tilde{z}$ must be unique. The statement is thus proven. The rest part of the proposition follows immediately from Proposition 4. Thus, Proposition 5 is proven. □

Proof of Proposition 7.

Proof. We first prove that $\theta_0^{d_o} < \theta_0^{m_o}$. Suppose not, then $\theta_0^{d_o} \geq \theta_0^{m_o}$. Note that $\theta_1^{d_o} - \theta_0^{d_o} = \theta_1^{m_o} - \theta_0^{m_o} = \sqrt{\frac{2k}{\gamma+3}}$, thus $\theta_1^{d_o} \geq \theta_1^{m_o}$. From Propositions 1 and 6, we have

$$z^{d_o}(\theta_1^{d_o}) = z^{m_o}(\theta_1^{m_o}) = \frac{\sqrt{2(\gamma+3)k}}{4}.$$

Since $\theta_1^{d_o} \geq \theta_1^{m_o}$, for $\theta \in [\theta_1^{m_o}, \theta_1^{d_o}]$, $z^{m_o'}(\theta) = \frac{\gamma+2}{2}$. This implies that $z^{m_o}(\theta_1^{d_o}) > z^{d_o}(\theta_1^{d_o})$.

Moreover, from (4.1), we have that for $\theta \in [\theta_1^{d_o}, 1]$,

$$\begin{aligned} z^{d_o'}(\theta) &= \frac{\gamma+2}{2} - \frac{1}{k}[(\gamma+1)\theta z^{d_o}(\theta) - z^{d_o^2}(\theta) - V^{d_o}(\theta)] \\ &= \frac{\gamma+2}{2} - \frac{1}{k}[S(z^{d_o}(\theta), \theta) - V^{d_o}(\theta)] \end{aligned}$$

In equilibrium, each school must gain positive profit for each type in the fully covered range, i.e., $S(z^{d_o}(\theta), \theta) > V^{d_o}(\theta)$ for $\theta > \theta_1^{d_o}$. Thus, we have for $\theta \in [\theta_1^{d_o}, 1]$,

$$z^{d_o'}(\theta) < z^{m_o'}(\theta) = \frac{\gamma+2}{2}.$$

Since $z^{m_o}(\theta_1^{d_o}) > z^{d_o}(\theta_1^{d_o})$, we have $z^{m_o}(1) > z^{d_o}(1)$. This contradicts the fact that $z^{m_o}(1) = z^{d_o}(1) = z^{fb}(1)$. Thus, we have $\theta_0^{d_o} < \theta_0^{m_o}$. This also implies that $\theta_1^{d_o} < \theta_1^{m_o}$.

We then prove that $z^{d_o}(\theta) > z^{m_o}(\theta)$ on $(\theta_0^{d_o}, 1)$. First, consider $\theta \in (\theta_0^{d_o}, \theta_1^{d_o}]$. Since on this interval $z^{m_o'}(\theta) = z^{d_o'}(\theta)$ and $\theta_0^{d_o} < \theta_0^{m_o} = \frac{\gamma+2}{2}$, we have $z^{d_o}(\theta) > z^{m_o}(\theta)$ for all $\theta \in (\theta_0^{d_o}, \theta_1^{d_o}]$. Second, consider $\theta \in (\theta_1^{d_o}, \theta_1^{m_o}]$. Since $z^{d_o}(\theta_1^{d_o}) = z^{m_o}(\theta_1^{m_o})$ and both $z^{d_o}(\theta)$ and $z^{m_o}(\theta)$ are increasing, we have $z^{d_o}(\theta) > z^{m_o}(\theta)$ for all $\theta \in (\theta_1^{d_o}, \theta_1^{m_o}]$. Finally, consider $\theta \in (\theta_1^{m_o}, 1]$. Due to the above analysis, we have $z^{d_o'}(\theta) < z^{m_o'}(\theta)$ on $(\theta_1^{m_o}, 1]$. It follows from $z^{m_o}(1) = z^{d_o}(1)$ that $z^{d_o}(\theta) > z^{m_o}(\theta)$ for all $\theta \in (\theta_1^{m_o}, 1)$. Thus, we have $z^{d_o}(\theta) > z^{m_o}(\theta)$ on $(\theta_0^{d_o}, 1)$.

Since $\theta_0^{d_o} < \theta_0^{m_o}$ and $z^{d_o}(\theta) > z^{m_o}(\theta)$ on $(\theta_0^{d_o}, 1)$, from Lemma 1, we have $V^{d_o}(\theta) > V^{m_o}(\theta)$ on $(\theta_0^{d_o}, 1)$. It is thus readily confirmed that more types, w.r.t. both horizontal and vertical types, are served in the market under duopoly. Thus, the proposition is proven. □

References

Armstrong, Mark, and John Vickers. 2001. "Competitive Price Discrimination". *RAND Journal of Economics*: 579–605.

Ellison, Glenn. 2005. "A Model of Add-on Pricing". *The Quarterly Journal of Economics* 120 (2): 585–637.

Gilbert, Richard J, and Carmen Matutes. 1993. "Product Line Rivalry with Brand Differentiation". *The Journal of Industrial Economics*: 223–240.

Johnson, Justin P, and David P Myatt. 2006. "On the Simple Economics of Advertising, Marketing, and Product Design". *American Economic Review* 96 (3): 756–784.

Lu, Zhuoran. 2018. "Selling Signals". Working Paper.

Manelli, Alejandro M, and Daniel R Vincent. 2006. "Bundling as an Optimal Selling Mechanism for a Multiple-Good Monopolist". *Journal of Economic Theory* 127 (1): 1–35.

Martimort, David, and Lars Stole. 2002. "The Revelation and Delegation Principles in Common Agency Games". *Econometrica* 70 (4): 1659–1673.

Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green. 1995. *Microeconomic theory.* Vol. 1. Oxford university press New York.

Maskin, Eric, and John Riley. 1984. "Monopoly with Incomplete Information". *The RAND Journal of Economics* 15 (2): 171–196.

McAfee, R Preston. 1993. "Mechanism Design by Competing Sellers". *Econometrica*: 1281–1312.

Mussa, Michael, and Sherwin Rosen. 1978. "Monopoly and Product Quality". *Journal of Economic theory* 18 (2): 301–317.

Rayo, Luis. 2013. "Monopolistic Signal Provision". *The BE Journal of Theoretical Economics* 13 (1): 27–58.

Rochet, Jean-Charles, and Lars A Stole. 2002. "Nonlinear Pricing with Random Participation". *The Review of Economic Studies* 69 (1): 277–311.

Seierstad, Atle, and Knut Sydsaeter. 1986. *Optimal Control Theory with Economic Applications.* Elsevier North-Holland, Inc.

Spence, Michael. 1973. "Job Market Signaling". *The Quarterly Journal of Economics*: 355–374.

Stole, Lars A. 1995. "Nonlinear Pricing and Oligopoly". *Journal of Economics & Management Strategy* 4 (4): 529–562.

Thanassoulis, John. 2004. "Haggling over Substitutes". *Journal of Economic theory* 117 (2): 217–245.

Verboven, Frank. 1999. "Product Line Rivalry and Market Segmentation—with an Application to Automobile Optional Engine Pricing". *The Journal of Industrial Economics* 47 (4): 399–425.

Villas-Boas, J Miguel, and Udo Schmidt-Mohr. 1999. "Oligopoly with Asymmetric Information: Differentiation in Credit Markets". *The RAND Journal of Economics*: 375–396.

Yang, Huanxing, and Lixin Ye. 2008. "Nonlinear Pricing, Market Coverage, and Competition". *Theoretical Economics* 3 (1): 123–153.

Ye, Lixin, and Chenglin Zhang. 2017. "Monopolistic Nonlinear Pricing with Consumer Entry". *Theoretical Economics* 12 (1): 141–173.