Competition, Reputation and Survival

Zhuoran Lu*

September 15, 2014

Abstract

This paper studies a duopoly exit game under a perfect good news learning process, in which each firm is unaware of any firm's quality but can observe both firms' reputations. A firm incurs flow costs when operating in the market; the firm's revenue depends on both its own reputation and the competitor's. When both firms have the same initial reputations, the pure strategy Nash equilibria are asymmetric with the firms exiting the market at different times; in the symmetric mixed strategy Nash equilibrium, both firms exit at an increasing exiting rate. In contrast, when the firms have different initial reputations, the unique subgame perfect Nash equilibrium requires that the firm with a higher reputation outlasts the firm with a lower reputation.

1 Introduction

For experience goods, the quality of the good is difficult to observe in advance, but can be ascertained upon consumption. Therefore, before making their purchase decision for an experience good, customers often search the reviews of the good's quality. For example, one will read customer reviews such as Yelp reviews when considering having dinner in a new restaurant. Thus, the reputation of the good's quality plays a significant role for a seller's revenue and profit. In particular, when a seller suffers a bad reputation for a relatively long time, it may consider exiting the market. However, when a seller is making such a decision, it should take into account not only its current and future profit flows, but also the possibility of its reputation raising in the future thereby achieving higher profits. Furthermore, if the seller is not a monopolist in the market, it may also care about its competitors' reputations and exit decisions. This is because the competitors' reputations have an immediate impact

*Department of Economics, UCLA. Email: zhuoranlu.econ@gmail.com.

on the seller's market share, and thus, they matter to the seller's profits; the competitors' exit decisions will affect the seller's future profitability if it chooses to stay, thereby affecting its opportunity cost of exiting.

In this paper, we study a duopoly exit game in which two firms are selling similar but differentiated goods. The quality of each good cannot be observed by anyone in the market, including consumers and both firms. Players learn the quality of a good through a perfect good news learning process under which a firm is convinced to be a high-quality producer if a good news occurs; otherwise its "reputation" will updates according to Bayes' rule. Each firm's revenue depends on not only its own reputation but also the competitor's, and is increasing in the former but decreasing in the latter. Moreover, at every instant of time each firm has to pay a fixed operating cost if it chooses to stay in the market. Therefore, a firm's profit can be negative when its reputation falls below some threshold.

We analyze the firms' exit decisions. The equilibrium depends critically on whether the firms' initial reputations are identical or not. If the firms start with the same reputations, the game can be reduced to a *Chicken Game* using Iterated Elimination of Strictly Dominated Strategies (IESDS). Consequently, there exist two asymmetric pure strategy Nash equilibria with the firms exiting at different times; in the symmetric mixed strategy equilibrium, the firms exit randomly at an increasing exiting rate within the support of time. On the other hand, if the firms' initial reputations are different, while there could still be asymmetric pure strategy equilibria, the unique subgame perfect Nash equilibrium requires that the firm with higher reputation always outlasts the other, leading to an appealing result. The underlying intuition is that the firm who can sustain more losses, namely the "stronger" one, can force the competitor to exit earlier.

1.1 Related Literatures

Our starting point is Ghemawat and Nalebuff (1985), in which two firms with different fixed capacity are competing in a declining industry. The main result of the paper is that in the unique subgame perfect Nash equilibrium, the firm with larger capacity thereby higher fixed cost will die earlier. Our model is richer in the sense that firms' flow profit is determined by a perfect good news learning process which provides firms an option value of reputational jumps. We also study the equilibria separately under different assumptions about the initial reputations. Thus, the model yields two additional asymmetric pure strategy Nash equilibria and a symmetric mixed strategy Nash equilibrium.

The exit decision is closely related to the models of patent races in the sense that continuing to operate in the market is similar to continuing to do research; exiting the market is comparable to abandoning the project. Fudenberg et al. (1983) show that the first firm that commits credibly to continuing research can force its rival to exit even if under the pessimistic conditions of continued competition. Analogously, in our model since the firms do not observe own qualities, the firm with a higher reputation can force its rival to exit even if the former is actually a low-quality firm but the latter is a high-quality firm. The uncertainty about own quality provides a higher-reputation firm with commitment power to stay, as the higher firm's expected future profit is higher than the lower firm's.

Our paper is also closely related to the exiting part of Board and Meyer-ter-Vehn (2014). In their paper, the monopolist exits definitely when its current losses is exactly compensated by the option value of a breakthrough. In our duopoly model, in addition, the option value from the competitor's exit also plays a significant role, leading to a multiplicity of equilibria.

1.2 Plan For The Paper

In the following section, we introduce the framework for the models we go on to consider. In Section 3, we conduct a preliminary analysis about equilibrium in some special circumstance. Section 4 is the main part of the paper where we characterize the equilibria separately under different assumptions about firms' initial reputations. In section 5, we discuss and conclude.

2 Model

Consider a market in which two long-lived and risk-neutral firms are selling differentiated experience goods. The quality of each firm's good, θ, is either high or low, i.e., $\theta \in \{H, L\}$, with $L = 0$ and $H = 1$. Both high- and low-quality firms incur zero cost in producing the goods. Time is continuous and infinite. At the beginning of every time t, each firm chooses whether to keep operation or to shut down. If the firm chooses to stay then it incurs a fixed operating cost k; otherwise the firm exits the market permanently and obtain 0 payoff thereafter. Every firm's quality are unobservable to both firms or the market.

Consumers learn about qualities through public *breakthroughs* which perfectly reveal a firm's high-quality. Given firm i's quality θ_i, $i = 1, 2$, public breakthroughs are generated according to two independent Poisson processes with the identical arrival rate $\lambda\theta$. Therefore, a high-quality firm receives breakthroughs with constant rate λ while a low-quality firm never enjoys breakthrough. Consumers' belief about firm i's high-quality at time t, x_{it} is termed the firm's *reputation* which is updated following the learning process. Upon receiving a breakthrough, since it perfectly reveals high-quality, firm i's reputation jumps to 1 and stays at 1 thereafter. For simplicity, we assume that once firm i exits the market, its reputation drops to 0 and stays there permanently. Since each firm has the same information as the market, firm i's belief about both firms' high-quality is also given by $\{x_{it}\}_{t\geq 0}$, $i = 1, 2$.

For simplicity, we do not model explicitly consumers' behavior; rather, we assume that firm i's instantaneous revenue at time t is given by a continuously differentiable function

$$g_i(x_{it}, x_{jt}), i = 1, 2.$$

Therefore, firm i's flow profit is given by

$$\pi_{it} := g_i(x_{it}, x_{jt}) - k. \tag{1}$$

We assume that for all dates $t \in [0, \infty]$,[1]

$$\frac{\partial g_i(x_{it}, x_{jt})}{\partial x_{it}} > 0, \tag{2a}$$

$$\frac{\partial g_i(x_{it}, x_{jt})}{\partial x_{jt}} < 0, \tag{2b}$$

$$g_i(1, 0) > g_i(1,\ 1) > k > g_i(0, 0) > g_i(0, 1). \tag{2c}$$

Conditions (2a) and (2b) state that the firm's flow profit is increasing in its own reputation but decreasing in the other's. Condition (2c) compares the operation cost k with the four boundary values of $g_i(x_{it}, x_{jt})$ with respect to (x_{it}, x_{jt}). In doing so, the reputation space $\{(x_{it}, x_{jt})\}$ is partitioned by k into the area of "making money" and that of "losing money". According to (2a) and (2b), firm i's flow profit is bounded by $[g_i(0, 1) - k, g_i(1, 0) - k]$.

Both firms discount future payoff at a common rate $r > 0$ and maximize the discounted present value. Furthermore, we assume that the firms have no access to saving or borrowing. The histories of firms' reputation and exit-decision are public information.

Suppose that both firms start business simultaneously with an initial reputation x_{i0}, $i = 1, 2$, respectively. We summarize the action as the repetition of the following stages:

(i) firm i observes reputation x_i^{t-} and x_j^{t-} up to but not including time t;

(ii) it decides whether to shut down or not;

(iii) if stay, it collects profit π_{it}.

[1] "∞" included because a firm can choose to stay in the market forever.

(iv) x_{it} and x_{jt} updates according to public signals.

2.1 Reputation Dynamics

We assume that every player in the game is Bayesian. If no signal arrives in $[t, t+\delta]$, then x_{it} updates according to Bayes' rule:

$$x_{it+\delta} = \frac{x_{it}(1-\lambda\delta)}{x_{it}(1-\lambda\delta)+1-x_{it}} + o(\delta). \tag{3}$$

The right hand side reflects the market's learning about quality based on the absence of a signal in $[t, t+\delta]$. In the limit as $\delta \to 0$, reputation x_{it} (in the absence of signals) is then governed by the ordinary differential equation (ODE) $\dot{x}_i = \eta(x_i)$, where the *reputational drift* is given by

$$\eta(x_i) = -\lambda x_i(1-x_i). \tag{4}$$

That is, in the absence of a breakthrough, firm i's reputation drifts downward due to adverse market belief. If there is a breakthrough, firm i's reputation jumps to $x_i = 1$ and stays at 1 thereafter.

Lemma 1. *For different initial levels of reputation $\hat{x}_0 > x_0$, we have $\hat{x}_t \geq x_t$ for any $t \geq 0$.*

Proof. For $t = 0$, Lemma 1 holds by assumption as $\hat{x}_0 > x_0$. Without any good news, the reputation is governed by (4), using method of Bernoulli Equation, denote $w = x^{-1}$ and rewrite (4) as

$$\dot{w} - \lambda w = -\lambda.$$

The solution to this ODE is given by

$$w_t = (A-1)e^{\lambda t} + 1,$$

for some constant A. Substituting $w = x^{-1}$ and the initial conditions $\hat{x}_0$, x_0, we have

$$\hat{x}_t = [(\frac{1}{\hat{x}_0} - 1)e^{\lambda t} + 1]^{-1} \text{ and } x_t = [(\frac{1}{x_0} - 1)e^{\lambda t} + 1]^{-1}.$$

Since $\hat{x}_0 > x_0$, thus $\hat{x}_t > x_t$ before the arrival of good news. At the time of jump, since both $\hat{x}_t$ and x_t jump to 1 and stay at 1 thereafter. Therefore $\hat{x}_t \geq x_t$ for any $t \geq 0$. □

2.2 Optimal Exit Time

In this subsection, we analyze a firm's optimal strategy which maximizes the firm's expected present value. The pure strategy of firm i is to choose an exit time $\tau_i \in [0, \infty]$; the mixed strategy is a probability distribution over the exit time $\Delta(\tau_i) \in [0, 1]$.

Based on the initial reputation x_{i0}, x_{j0}, firms can infer the distribution of the path of future reputations and thereby that of the future flow profits. We write $\mathbb{E}^{x_{i0},x_{j0}}$ as firm i's expectation under this measure and assume that x_{i0}, x_{j0} satisfy that both firms are willing to operate at time 0.

Since firm i's payoff is discontinuous upon firm j's exit if the former outlasts the latter, it is tempting to decompose firm i's expected present value into two parts. Let z be the last date such that *both* firms are operating in the market. Denote by $P_i(z, 0)$ firm i's expected present value at time 0; denote by $C_i(z, 0)$ firm i's expected profits (or losses) discounted back to date 0 from operating over the period $[0, z)$; denote by $V_i(z, 0)$ firm i's expected discounted profits (or losses) from operation after its competitor exits at z. Given firm i's strategy τ_i, if z is less than τ_i, then we have

$$C_i(z, 0) = \mathbb{E}^{x_{i0},x_{j0}} \int_0^z e^{-rs}[g_i(x_{is}, x_{js}) - k]ds, \tag{5a}$$

$$V_i(z, 0) = \mathbb{E}^{x_{i0},x_{j0}} \int_z^{\tau_i} e^{-rs}[g_i(x_{is}, 0) - k]ds, \tag{5b}$$

$$P_i(z, 0) = C_i(z, 0) + V_i(z, 0). \tag{5c}$$

If z is greater than or equal to τ_i, then $V_i(z, 0) = 0$. Therefore, given firm j's exit time τ_j, the optimal exit time of firm i, τ_i^*, maximizes $P_i(min\{\tau_i^*, \tau_j\}, 0)$.

3 Preliminary Analysis

In this section, we start equilibrium analysis by focusing on some special circumstances. As a reference point, we first analyze the equilibrium strategies when both firms have received a breakthrough.

Suppose at time t, $x_{it} = x_{jt} = 1$, if firm i chooses to stay, then its future flow profit is bounded below by $g_i(1, 1) - k$ which is strictly positive according to the model assumption. That is, keeping operating permanently is a dominant strategy for both firms. Thus, the unique (perfect) Nash equilibrium in this situation is that $\tau_i^* = \tau_j^* = \infty$. Let $J_i(1, 1)$ be the equilibrium present value of firm $i = 1$, 2, where the first and second argument of $J_i(\cdot, \cdot)$ are firm i and j's reputation, respectively. Since the flow profit is fixed by $g_i(1,1) - k$, we have

$$J_i(1,1) = \frac{g_i(1,1) - k}{r}. \tag{6}$$

Now suppose that only one firm has received breakthrough. It is without loss of generality to assume that at time t, $x_{it} < 1$ and $x_{jt} = 1$. Analogously, firm j's dominant strategy is that $\tau_j^* = \infty$. Therefore, in equilibrium, firm i's expected present value is $P_i(\tau_i^*, t)$. Upon receiving a breakthrough, firm i's optimal strategy is $\tau_i^* = \infty$; thereby its continuation value in equilibrium equals $J_i(1,1)$. Truncate $P_i(\tau_i^*, t)$ at firm i's first breakthrough we have

$$P_i(\tau_i^*, t) = \int_t^{\tau_i^*} e^{-\int_t^s (r + x_{iu}\lambda) du} [g_i(x_{is}, 1) - k + x_{is}\lambda J_i(1,1)] ds. \tag{7}$$

Note that $g_i(x_{is}, 1) - k + x_{is}\lambda J_i(1,1)$ is strictly increasing in x_{is}, and by (4) we know that in the absence of breakthrough, x_{is} is continuous and strictly decreasing in time s, thus $g_i(x_{is}, 1) - k + x_{is}\lambda J_i(1,1)$ is continuous and strictly decreasing in time s without news. Moreover, for sufficiently small x_{is}, $g_i(x_{is}, 1) - k + x_{is}\lambda J_i(1,1) < 0$, satisfying the single-crossing property. Thus, firm i's optimal exit time τ_i^* satisfies the condition below.[2]

$$g_i(x_{i\tau_i^*}, 1) - k + x_{i\tau_i^*}\lambda J_i(1,1) = 0. \tag{8}$$

Equation (8) indicates that at the exit time firm i makes a loss, which is exactly compensated by the option value of staying in the market and the potential of enjoying the benefit of a good signal. Given the monotonicity of the left-hand side (LHS) of (8), the optimal strategy is a singleton. Therefore, the unique (perfect) Nash equilibrium here is that $\tau_j^* = \infty$ and τ_i^* satisfies (8). Denote firm i's equilibrium continuation value at time t by $J_i(x_{it}, 1)$.

Finally, we characterize firm i's optimal strategy when it is the only firm serving the market. Similarly, if firm i has received a breakthrough, the optimal strategy is $\tau_i^* = \infty$ and the optimal present value is given by

$$J_i(1,0) =: \frac{g_i(1,0) - k}{r}; \tag{9}$$

otherwise, analogous to the above situation, firm i's optimal strategy τ_i^* satisfies

$$g_i(x_{i\tau_i^*}, 0) - k + x_{i\tau_i^*}\lambda J_i(1,0) = 0. \tag{10}$$

[2]See the proof of Theorem 3 in Board and Meyer-ter-Vehn (2014).

4 Equilibrium before Exit and Breakthrough

The last but the most interesting circumstance is that both firms are present and no one has received a breakthrough. Consider any time t by which no firm has exited, we start by assuming that firms' reputation are identical.

4.1 $x_{it} = x_{jt}$

Suppose at time t, $x_{it} = x_{jt} < 1$. We first characterize the pure strategy Nash equilibria by using the method of *Iterated Elimination of Strictly Dominated Strategies* (IESDS).

Given firm j's exit time, τ_j, if firm i exits at date τ_i, then according to Section 2.2 its payoff equals

$$P_i(z,t) = C_i(z,t) + V_i(z,t). \tag{11}$$

where $z = min\{\tau_i, \tau_j\}$. Truncate $C_i(z,t)$ and $V_i(z,t)$, respectively, at the first arrival of breakthrough:

$$\begin{aligned} C_i(z,t) = \int_t^z & e^{-\int_t^s (r+x_{iu}\lambda+x_{ju}\lambda)du}[g_i(x_{is},x_{js}) - k \\ & + x_{is}\lambda J_i(1,x_{js}) + x_{js}\lambda J_i(x_{is},1)]ds, \end{aligned} \tag{12a}$$

$$V_i(z,t) = \int_z^{\tau_i} e^{-\int_t^s (r+x_{iu}\lambda)du}[g_i(x_{is},0) - k + x_{is}\lambda J_i(1,0)]ds. \tag{12b}$$

where $J_i(1,x_{js})$ is the counterpart of $J_i(x_{is},1)$ when both firms are present and only firm i has received a breakthrough.

Lemma 2. *$J_i(x_{it},1)$ and $J_i(1,x_{jt})$ are both continuous in time t.*

Proof. We first prove the continuity of $J_i(x_{it},1)$. Note that $J_i(x_{it},1)$ is bounded by the interval $[(g_i(0,1)-k)/r, (g_i(1,1)-k)/r]$ for all x_{it}, as the flow profit is bounded by the interval $[g_i(0,1)-k, g_i(1,1)-k]$. Truncate $J_i(x_{it},1)$ at either firm i's first breakthrough or t', whichever is earlier, we have

$$\begin{aligned} J_i(x_{it},1) = & \int_t^{t'} e^{-\int_t^s (r+x_{iu}\lambda)du}[g_i(x_{it},1) - k + x_{is}\lambda J_i(1,1)]ds \\ & + e^{-\int_t^{t'} (r+x_{iu}\lambda)du} J_i(x_{it'},1). \end{aligned}$$

The integral is of order $t'-t$, $J_i(x_{it'},1)$ is bounded and $e^{-\int_t^{t'} (r+x_{iu}\lambda)du} \in [1-(r+\lambda)(t'-t), 1]$.

This implies that $J_i(x_{it}, 1)$ is Lipschitz continuous in t for some Lipschitz constant. The proof for the continuity of $J_i(1, x_{jt})$ is analogous. □

Lemma 2 means that the integrand of (12a) and (12b) are continuous. Consequently, the integrand of (12b) has a single-crossing property while that of (12a) doesn't have generally. To simplify the analysis, we introduce the following assumption and employ it throughout the subsequent parts of the paper.

Assumption 1. *As long as x_{is}, $x_{js} \in (0, 1)$, $g_i(x_{is}, x_{js})$ satisfies that*

$$g_i(x_{is}, x_{js}) - k + x_{is}\lambda J_i(1, x_{js}) + x_{js}\lambda J_i(x_{is}, 1)$$

is decreasing in time s.

Intuitively, if x_j has sufficiently small impacts on firm i's flow profit, then the drift of $g_i(x_{is}, x_{js}) - k + x_{is}\lambda J_i(1, x_{js}) + x_{js}\lambda J_i(x_{is}, 1)$ is dominantly determined by that of x_{is} as long as $x_{is} \in (0, 1)$. Since $g_i(x_{is}, x_{js}) - k + x_{is}\lambda J_i(1, x_{js}) + x_{js}\lambda J_i(x_{is}, 1)$ is increasing in x_{is} and the latter is decreasing in time s in the absence of breakthrough, Assumption 1 is feasible. Note that $g_i(x_{is}, x_{js}) - k + x_{is}\lambda J_i(1, x_{js}) + x_{js}\lambda J_i(x_{is}, 1) < 0$ for sufficiently large s (in the absence of breakthrough), thus given Assumption 1, the integrand of (12a) is single-crossing.

Then, define dates $\underline{\tau}$ and $\bar{\tau}$ such that $\underline{\tau}$ and $\bar{\tau}$ satisfy

$$g_i(x_{i\underline{\tau}}, x_{j\underline{\tau}}) - k + x_{i\underline{\tau}}\lambda J_i(1, x_{j\underline{\tau}}) + x_{j\underline{\tau}}\lambda J_i(x_{i\underline{\tau}}, 1) = 0, \tag{13a}$$

$$g_i(x_{i\bar{\tau}}, 0) - k + x_{i\bar{\tau}}\lambda J_i(1, 0) = 0. \tag{13b}$$

That is, $\underline{\tau}$ is the time when firm i is break-even in duopoly and $\bar{\tau}$ is the time when firm i is break-even in monopoly.

It is worth noticing that $J_i(x_{i\underline{\tau}}, 1) = 0$. This is because if firm j receives a breakthrough at $\underline{\tau}$, then according to Section 3 it will never exit in equilibrium, thus by Assumption 1 firm i's future flow payoffs are bounded above by $g_i(x_{i\underline{\tau}}, 1) - k + x_{i\underline{\tau}}\lambda J_i(1, 1)$ which is less than $g_i(x_{i\underline{\tau}}, x_{j\underline{\tau}}) - k + x_{i\underline{\tau}}\lambda J_i(1, x_{j\underline{\tau}}) + x_{j\underline{\tau}}\lambda J_i(x_{i\underline{\tau}}, 1) = 0$. Therefore, in equilibrium, firm i exits immediately if firm j receives a breakthrough at time $\underline{\tau}$, meaning that $J_i(x_{i\underline{\tau}}, 1) = 0$. As a result, we can rewrite (13a) as

$$g_i(x_{i\underline{\tau}}, x_{j\underline{\tau}}) - k + x_{i\underline{\tau}}\lambda J_i(1, x_{j\underline{\tau}}) = 0. \tag{13c}$$

Since $g_i(x_{it}, x_{jt}) - k + x_{it}\lambda J_i(1, x_{jt}) < g_i(x_{it}, 0) - k + x_{it}\lambda J_i(1, 0)$ for all $t > 0$, by the single-crossing property, we have that $\underline{\tau} < \bar{\tau}$.

We now show that any strategy $\tau_i \in [t, \underline{\tau}) \cup (\bar{\tau}, \infty]$ is strictly dominated. Note that the monopolist always has a higher payoff since the flow profits $g_i(x_{it}, x_{jt}) - k < g_i(x_{it}, 0) - k$ for all t; for any $s > \bar{\tau}$, firm i's flow payoff is bounded above by $g_i(x_{is}, 0) - k + x_{is}\lambda J_i(1, 0)$ which is always negative by the single-crossing property. If firm i exits at $\tau_i > \bar{\tau}$, then it suffers the losses from operation during $[\bar{\tau}, \tau_i]$ for just nothing. This means that any $\tau_i > \bar{\tau}$ is dominated by $\bar{\tau}$. On the other hand, since for any $s < \underline{\tau}$ the flow payoff is positive by Assumption 1, if firm i exits at $\tau_i < \underline{\tau}$ then it misses the profits of operating during $[\tau_i, \underline{\tau}]$. Therefore, any $\tau_i < \underline{\tau}$ is dominated by $\underline{\tau}$. This proves the claim. Furthermore, by symmetry, any $\tau_j \in [t, \underline{\tau}) \cup (\bar{\tau}, \infty]$ is also strictly dominated for firm j.

Then given that $\tau_j \in [\underline{\tau}, \bar{\tau}]$ in equilibrium, any $\tau_i \in (\underline{\tau}, \bar{\tau})$ is strictly dominated for firm i since if $\tau_i < \tau_j$, firm i should exit earlier at $\underline{\tau}$ to avoid losses during $[\underline{\tau}, \tau_i]$; if $\tau_i > \tau_j$, firm i would better off exiting later at $\bar{\tau}$ to gain the profit during $[\tau_i, \bar{\tau}]$; if $\tau_i = \tau_j$, either $\underline{\tau}$ or $\bar{\tau}$ will provide higher payoff. Again by symmetry, any $\tau_j \in (\underline{\tau}, \bar{\tau})$ is strictly dominated for firm j. Finally, the stratgies surviving from IESDS are only $\underline{\tau}$ and $\bar{\tau}$ for both firms.

For simplicity, let $t = \underline{\tau}$, it is heuristic to present the game in a normal form,

		firm j: $\underline{\tau}$		firm j: $\bar{\tau}$	
firm i	$\underline{\tau}$	0	0	0	$V_i(\underline{\tau}, \bar{\tau})$
	$\bar{\tau}$	$V_j(\underline{\tau}, \bar{\tau})$	0	$C_i(\underline{\tau}, \bar{\tau})$	$C_j(\underline{\tau}, \bar{\tau})$

where

$$V_i(\underline{\tau}, \bar{\tau}) = V_j(\underline{\tau}, \bar{\tau}) = \int_{\underline{\tau}}^{\bar{\tau}} e^{-\int_{\underline{\tau}}^{s}(r + x_{iu}\lambda)du}[g_i(x_{is}, 0) - k + x_{is}\lambda J_i(1, 0)]ds$$
$$>0, \tag{14a}$$
$$C_i(\underline{\tau}, \bar{\tau}) = C_j(\underline{\tau}, \bar{\tau}) = \int_{\underline{\tau}}^{\bar{\tau}} e^{-\int_{\underline{\tau}}^{s}(r + x_{iu}\lambda + x_{ju}\lambda)du}[g_i(x_{is}, x_{js}) - k$$
$$+ x_{is}\lambda J_i(1, x_{js}) + x_{js}\lambda J_i(x_{is}, 1)]ds$$
$$<0. \tag{14b}$$

The payoff matrix states that there is one and only one firm can have positive payoff during the time $[\underline{\tau}, \bar{\tau}]$. Therefore, *coexit* cannot happen in the equilibrium and the set of pure strategy Nash equilibria is that

$$\{(\tau_i = \underline{\tau}, \tau_j = \bar{\tau}), (\tau_i = \bar{\tau}, \tau_j = \underline{\tau})\}.$$

We summarize the pure strategy equilibria by the following position.

Proposition 1. *Given Assumption 1, if both firms have not exited by time* $t < \underline{\tau}$ *and* $x_{it} = x_{jt} < 1$*, there exist two pure strategy Nash equilibria: in the first, firm* i *exits earlier than firm* j *such that* $\tau_i^* = \underline{\tau}$ *and* $\tau_j^* = \bar{\tau}$*; in the second, firm* j *exits earlier such that* $\tau_j^* = \underline{\tau}$ *and* $\tau_i^* = \bar{\tau}$*, where* $\underline{\tau}$ *and* $\bar{\tau}$ *are defined by (13a) and (13b) respectively.*

The possibility of mixed strategy Nash equilibrium remains open. Here, we only look for symmetric mixed strategy equilibrium. Denote $F(s)$ as the cumulative distribution function of firms' exit time s in the equilibrium. Since any $s \in [t, \underline{\tau}) \cup (\bar{\tau}, \infty]$ is strictly dominated for both firms, the support of $F(s)$ must belongs to $[\underline{\tau}, \bar{\tau}]$.

Lemma 3. *There is no gap or atom in* $F(s)$ *on the interval* $[\underline{\tau}, \bar{\tau}]$*.*

Proof. In a mixed strategy equilibrium, firms are indifferent among all exit times. Suppose there exists an interval $[a, b] \subset [\underline{\tau}, \bar{\tau}]$ such that $F(s)$ are constant on $[a, b]$, meaning that firm j will exit with 0 probability during $[a, b]$, then firm i will strictly prefer exiting at a to b because it can avoid losses from operation during $[a, b]$. This induces a contradiction with the indifference condition. Therefore, there is no gap.

Then by the indifference condition, we conclude that an atom can only appears at the very beginning of the domain. Suppose there is an atom at $\underline{\tau}$, meaning that both firm will exit at $\underline{\tau}$ with positive probability, then one firm can be better off by moving the atom to $\bar{\tau}$, leading to a profitable deviation. Therefore, there is no atom. □

Lemma 3 implies that $F(s)$'s support is $[\underline{\tau}, \bar{\tau}]$ and it is differentiable. Let $f(s)$ be the probability density function and define the *hazard rate* to be

$$H(s) := \frac{f(s)}{1 - F(s)}. \tag{15}$$

Without loss of generality, let the current time be $\underline{\tau}$. Denote by $M_i(t, \underline{\tau})$ firm i's expected present value at $\underline{\tau}$ if it chooses to exit at some time $t \in [\underline{\tau}, \bar{\tau}]$. Truncate $M_i(t, \underline{\tau})$ at the first breakthrough, and firm j's exit time, whichever is earlier:

$$M_i(t, \underline{\tau}) = \int_{\underline{\tau}}^{t} e^{-\int_{\underline{\tau}}^{s}(r+x_{iu}\lambda+x_{ju}\lambda+H(u))du}[g_i(x_{is}, x_{js}) - k + x_{is}\lambda J_i(1, x_{js}) + x_{js}\lambda J_i(x_{is}, 1) + H(s)V_i(s, s)]ds, \tag{16a}$$

where

$$V_i(s, s) = \int_{s}^{\bar{\tau}} e^{-\int_{s}^{v}(r+x_{iu}\lambda)du}[g_i(x_{iv}, 0) - k + x_{iv}\lambda J_i(1, 0)]dv. \tag{16b}$$

The term $H(s)V_i(s, s)$ in (16a) is the expected value appreciation due to the competitor's exit. By indifference, we have $M_i(t, \underline{\tau}) = M_i(\underline{\tau}, \underline{\tau}) = 0$ for any $t \in [\underline{\tau}, \bar{\tau}]$. This implies that in the equilibrium once firm j receives a breakthrough earlier than firm i, the latter will immediately exit upon arrival of the news for the same reason as above. Therefore, we have $J_i(x_{is}, 1) = 0$ for $s \in [\underline{\tau}, t]$. Similarly, firm j will exit upon arrival of breakthrough if firm i receives it earlier, thus we have $J_i(1, x_{js})$ equals $J_i(1, 0)$ given by (9). Substituting, we have

$$M_i(t, \underline{\tau}) = \int_{\underline{\tau}}^{t} e^{-\int_{\underline{\tau}}^{s}(r+x_{iu}\lambda+x_{ju}\lambda+H(u))du}[g_i(x_{is}, x_{js}) - k + x_{is}\lambda J_i(1, 0) + H(s)V_i(s, s)]ds. \tag{16c}$$

By a similar argument as in the proof of Lemma 2, we conclude that $V_i(s, s)$ is continuous in time s. This means that the integrand of the integration in (16c) is continuous in time s. Since $M_i(t, \underline{\tau}) = 0$ for any $t \in [\underline{\tau}, \bar{\tau}]$, for any $s \in [\underline{\tau}, \bar{\tau})$,

$$g_i(x_{is}, x_{js}) - k + x_{is}\lambda J_i(1, 0) + H(s)V_i(s, s) = 0. \tag{17}$$

Since $V_i(s, s) > 0$ for $s < \bar{\tau}$, we derive that

$$H(s) = \frac{k - g_i(x_{is}, x_{js}) - x_{is}\lambda J_i(1, 0)}{V_i(s, s)}. \tag{18}$$

Note that for $s \in [\underline{\tau}, \bar{\tau})$, the numerator is positive, so $H(s)$ is well-defined.

Lemma 4. *$V_i(s, s)$ is decreasing in s on $[\underline{\tau}, \bar{\tau})$.*

Proof. We first prove that $V_i(s, s)$ is increasing in the initial reputation x_{is}. Rewrite $V_i(s, s)$ as a function of x_{is}, $V_i(x_{is})$. Consider two hypothetical firms, "high" and "low" with different levels of initial reputation $\hat{x}_{is} > x_{is}$. Let τ_i^* be the optimal exiting time for the low firm and $\hat{\tau}_i^*$ be the optimal exiting time for the high firm. Suppose the high firm exits at the same time τ_i^*. Note that this is generally not the optimal exiting time for the high firm. By Lemma 1 we have $\hat{x}_{it} \geq x_{it}$ for any $t \geq s$, and thus,

$$\begin{aligned}
V_i(\hat{x}_{is}) &= \mathbb{E}^{\hat{x}_{is}}\left[\int_0^{\hat{\tau}_i^*} e^{-rt}(g_i(\hat{x}_{it}, 0) - k)dt\right] \\
&\geq \mathbb{E}^{\hat{x}_{is}}\left[\int_0^{\tau_i^*} e^{-rt}(g_i(\hat{x}_{it}, 0) - k)dt\right] \\
&\geq \mathbb{E}^{x_{is}}\left[\int_0^{\tau_i^*} e^{-rt}(g_i(x_{it}, 0) - k)dt\right] \\
&= V_i(x_{is}).
\end{aligned}$$

Therefore $V_i(s,s)$ is increasing in x_{is}. Since in the absence of breakthrough, x_{is} is decreasing in time s thereby $V_i(s,s)$ is decreasing in time s. □

Lemma 4 implies that $H(s)$ is increasing in $s \in [\underline{\tau}, \bar{\tau})$ because the numerator of $H(s)$ is increasing in time s by Assumption 1. Hence, upon surviving at time s, firm j is more likely to exit than the last instant. The intuition is that as time passed by, the losses firm j burdened is increasing but the gain from the competitor's exit is decreasing, so staying is less attractive. Finally, we summarize the equilibrium by the proposition below.

Proposition 2. *Given Assumption 1, if both firms have not exited by time $t < \underline{\tau}$ and $x_{it} = x_{jt} < 1$, the symmetric mixed strategy Nash equilibrium is such that upon surviving time $s \in [\underline{\tau}, \bar{\tau})$, each firm exits in the immediate dt time with an increasing probability $H(s)dt$, where $H(s)$ is defined by (18).*

4.2 $x_{it} \neq x_{jt}$

Without loss of generality, assume $x_{it} < x_{jt} < 1$. Define the dates $\underline{\tau}_i$, $\underline{\tau}_j$, $\bar{\tau}_i$ and $\bar{\tau}_j$ such that

$$g_i(x_{i\underline{\tau}_i}, x_{j\underline{\tau}_i}) - k + x_{i\underline{\tau}_i}\lambda J_i(1, x_{j\underline{\tau}_i}) = 0, \tag{19a}$$

$$g_j(x_{j\underline{\tau}_j}, x_{i\underline{\tau}_j}) - k + x_{j\underline{\tau}_j}\lambda J_j(x_{j\underline{\tau}_j}, 1) = 0, \tag{19b}$$

$$g_i(x_{i\bar{\tau}_i}, 0) - k + x_{i\bar{\tau}_i}\lambda J_i(1, 0) = 0, \tag{19c}$$

$$g_j(x_{j\bar{\tau}_j}, 0) - k + x_{j\bar{\tau}_j}\lambda J_j(0, 1) = 0. \tag{19d}$$

That is, $\underline{\tau}_i$ and $\underline{\tau}_j$ are respective firm's break-even time when both firms are present; $\bar{\tau}_i$ and $\bar{\tau}_j$ are respective firm's break-even time when the firm is the only one serving the market. Given

Assumption 1 and the monotonicity of the monopoly payoff, we know that in equilibrium, firm i and j will operate up to at earliest $\underline{\tau}_i$ and $\underline{\tau}_j$, and at latest $\bar{\tau}_i$ and $\bar{\tau}_j$ respectively. By Lemma 1 and since $x_{it} < x_{jt} < 1$, we have that for any time $s > t$, $x_{is} < x_{js}$ in the absence of breakthrough. Thus. (19a)-(19b) imply that $x_{i\underline{\tau}_i} > x_{j\underline{\tau}_j}$. Furthermore (19c)-(19d) imply that $x_{i\bar{\tau}_i} = x_{j\bar{\tau}_j}$. Since $x_{it} < x_{jt}$, we have

$$\underline{\tau}_i < \underline{\tau}_j \text{ and } \bar{\tau}_i < \bar{\tau}_j.$$

Obviously, if $\bar{\tau}_i \leq \underline{\tau}_j$, the unique (perfect) Nash equilibrium is that $\tau_i = \underline{\tau}_i$ and $\tau_j = \bar{\tau}_j$ because firm j will only exits after $\underline{\tau}_j$ regardless of firm i's exit time; thus it is optimal for firm i to exit at $\underline{\tau}_i$—the time when it is break-even given both firms are present—and in turn, firm j's optimal strategy is that $\tau_j = \bar{\tau}_j$—the time when it is break-even as a monopolist. The previous situation in which one firm's reputation has jumped to 1 is just a special case of the current situation.

If in contrast $\bar{\tau}_i > \underline{\tau}_j$, besides the above equilibrium, there can be another pure strategy Nash equilibrium that firm i exits at $\bar{\tau}_i$ and firm j exits at $\underline{\tau}_j$, and the possibility of mixed strategy equilibrium remains open. Intuitively, when $\bar{\tau}_i$ is sufficiently close to $\bar{\tau}_j$, firm i can compel firm j to exit at time $\underline{\tau}_j$ by threatening to stay up to $\bar{\tau}_i$ because firm j's monopoly payoff is not enough for it to sustain the losses.

However, the second equilibrium is not as appealing as the first one in the sense that firm j is "stronger" than firm i left alone, it remains profitable longer than firm i; that is, the Nash equilibrium that firm j exits earlier is not a perfect equilibrium. If firm j "trembles" and fails to leave immediately, $\tau_i = \bar{\tau}_i$ is no longer a optimal strategy for firm i.

Proposition 3. *Given Assumption 1, if both firms have not exited by time $t < \underline{\tau}_i$ and $x_{it} < x_{jt} < 1$, the unique perfect Nash equilibrium is that firm i exits earlier than firm j such that $\tau_i^* = \underline{\tau}_i$ and $\tau_j^* = \bar{\tau}_j$, where $\underline{\tau}_i$ and $\bar{\tau}_j$ are defined by (19a) and (19d) respectively.*

Proof. We prove by Backward Induction. Note that exiting after $\bar{\tau}_i$ is strictly dominated by exiting at $\bar{\tau}_i$, so firm i will exit by the time $\bar{\tau}_i$ in the equilibrium. Define time t_1 to be the first date such that firm j is willing to sustain losses over the period $[t_1, \bar{\tau}_i)$ to reap monopoly payoffs over $[\bar{\tau}_i, \bar{\tau}_j)$:[3]

$$P_j(\bar{\tau}_i, t_1) = C_j(\bar{\tau}_i, t_1) + V_j(\bar{\tau}_i, t_1) \equiv 0. \tag{20}$$

[3] $C_j(\bar{\tau}_i, t_1) = \int_{t_1}^{\bar{\tau}_i} e^{-\int_t^s (r + x_{iu}\lambda + x_{ju}\lambda)du}[g_j(x_{js}, x_{is}) - k + x_{is}\lambda J_j(1, x_{js}) + x_{js}\lambda J_j(x_{is}, 1)]ds$, $V_j(\bar{\tau}_i, t_1) = \int_{t_1}^{\bar{\tau}_i} e^{-\int_t^s (r + x_{ju}\lambda)du}[g_j(x_{js}, x_{is}) - k + x_{js}\lambda J_j(0, 1)]ds$. The former is negative and the latter is positive by the definition of $\bar{\tau}_i$ and t_1.

Since $\bar{\tau}_i$ is the latest time that firm i can have a positive payoff, firm j's most pessimistic belief is that firm i keeps operating until time $\bar{\tau}_i$. Even if with such belief, firm j which survives until time t_1 will choose to remain active up to time $\bar{\tau}_j$. Thus, if firm j manages to stay in the market until time t_1, firm i cannot make credible threat to prevent firm j from surviving until $\bar{\tau}_j$. Since the continued operation of both firms implies losses for firm i over $[t_1, \bar{\tau}_i)$, firm i's optimal strategy on reaching time t_1 is to exit immediately.

Proceed the above argument recursively. Calculate the earliest date t_2 such that firm j is break-even operating over $[t_2, \bar{\tau}_j)$:

$$P_j(t_1, t_2) = C_j(t_1, t_2) + V_j(t_1, t_2) \equiv 0. \tag{21}$$

Since firm j finds it optimal to remain in the market until $\bar{\tau}_j$ upon surviving until time t_2, firm i will find it optimal to exit immediately upon reaching time t_2. Then, calculate t_3 by the fact that firm i must exit at the very latest by time t_2. This process continues until the last date t_N at which firm i will immediately exit, so we have a sequence of time intervals $\{[t_n, t_{n-1})\}_{n\geq 2}$. To complete the proof we have to show that $t_N \leq \underline{\tau}_j$, this is guaranteed by the following lemma.

Lemma 5. *The length of the interval $[t_n, t_{n-1})$ is increasing in n as long as $t_n \geq \underline{\tau}_j$.*

Proof. We first conclude that $V_j(t_{n-1}, t_n)$ is increasing in n by the same reason of the proof in Lemma 4. Consequently, we have that $C_j(t_{n-1}, t_n)$ is decreasing in n by the break-even conditions of firm j. Truncate $C_j(t_{n-1}, t_n)$ at the first breakthrough,

$$\begin{aligned} C_j(t_{n-1}, t_n) = \int_{t_n}^{t_{n-1}} e^{-\int_{t_n}^{s}(r + x_{iu}\lambda + x_{ju}\lambda)du} [g_j(x_{js}, x_{is}) - k \\ + x_{is}\lambda J_j(1, x_{js}) + x_{js}\lambda J_j(x_{is}, 1)]ds. \end{aligned}$$

The exponent of the integration is positive and increasing in t_n; $g_j(x_{js}, x_{is}) - k + x_{is}\lambda J_j(1, x_{js}) + x_{js}\lambda J_j(x_{is}, 1)$ is decreasing in time s by Assumption 1. Thus, the integrand of $C_j(t_n, t_{n+1})$ is uniformly greater than that of $C_j(t_{n-1}, t_n)$. If $t_{n+1} \geq \underline{\tau}_j$, then the integrand of both $C_j(t_{n-1}, t_n)$ and $C_j(t_n, t_{n+1})$ is negative everywhere in the domain by the definition of $\underline{\tau}_j$. Since $C_j(t_n, t_{n+1}) < C_j(t_{n-1}, t_n) < 0$, we must have the length of $[t_{n+1}, t_n)$ is greater than that of $[t_n, t_{n-1})$. Therefore, we prove that the length of $[t_n, t_{n-1})$ is increasing in n. □

The idea of Lemma 5 is that dating back, firm j is willing to suffer a longer period of losses since the monopoly payoff $V_j(t_{n-1}, t_n)$ is increasing, meaning that the step back is larger and larger. Thus, there must exist some $[t_{n^*}, t_{n^*-1}) \in \{[t_n, t_{n-1})_{n\geq 2}\}$ that covers $\underline{\tau}_j$ and $P_j(t_{n^*-1}, \underline{\tau}_j) \geq 0$. This means that firm j can credibly commit to keep operating over $[\underline{\tau}_j, \bar{\tau}_j)$, and thus, firm i finds it optimal to exit at time $\underline{\tau}_i$.

Finally, note that the above argument also proves the non-existence of a mixed strategy Nash equilibrium, as on each time interval firm j will stay with probability 1; thereby firm i will not randomize its exit time. □

While there are multiple Nash equilibria in the exit game with asymmetric reputations, there exists a unique subgame perfect equilibrium in which the "stronger" firm always outlasts the inferior. In this sense, the equilibrium is efficient.

It is worth noting that the main result relies on the assumption that firms have no access to saving or borrowing. If firms' capabilities of borrowing are different, exit will occur sooner if the lower-reputation firm's capital constraint becomes binding first. The key idea is the same; that is, the one who can sustain more losses can force its competitor to exit earlier.

5 Discussion and Conclusion

In this paper, we studied an duopoly exit game under a perfect good news learning process. In the model, the firms' actual qualities cannot be observed by anyone, and their payoffs are determined by both firms' reputations. We showed that with identical paths of firms' reputations, the pure strategy Nash equilibria are asymmetric with the firms exit at different times; in the symmetric mixed strategy equilibrium firms exit at an increasing exiting rate. In contrast, if firms' reputations are different, the unique subgame perfect Nash equilibrium requires that the firm with a higher reputation always outlasts the other.

An natural and interesting extension of this model might be that ceteris paribus, firms know their own quality. Bar-Isaac (2003) shows that in a monopoly market, the equilibrium is such that a quality-informed high-quality firm never exit while an informed low-quality firm play mixed strategy at the exiting point. Due to the mechanism of war of attrition, the equilibrium in the duopoly model might differ a lot. For example, a high-quality firm, though its expected rate of news arrival is constant, doesn't necessarily always stay, especially when it is competing with another high-quality firm and its reputation is sufficiently low; on the other hand, a low-quality firm knowing that it never enjoy a breakthrough might outlast a high-quality firm by strategic play. In general, more asymmetric equilibria could appear no matter whether firms' qualities are identical or not. The equilibrium of this case remains to be studied in the future.

References

Bar-Isaac, Heski. 2003. “Reputation and Survival: learning in a dynamic signalling model”. *The Review of Economic Studies* 70 (2): 231–251.

Board, Simon, and Moritz Meyer-ter-Vehn. 2014. “A reputational theory of firm dynamics”. *Unpublished paper, UCLA.[430].*

Fudenberg, Drew, et al. 1983. “Preemption, leapfrogging and competition in patent races”. *European Economic Review* 22 (1): 3–31.

Ghemawat, Pankaj, and Barry Nalebuff. 1985. “Exit”. *The RAND Journal of Economics*: 184–194.